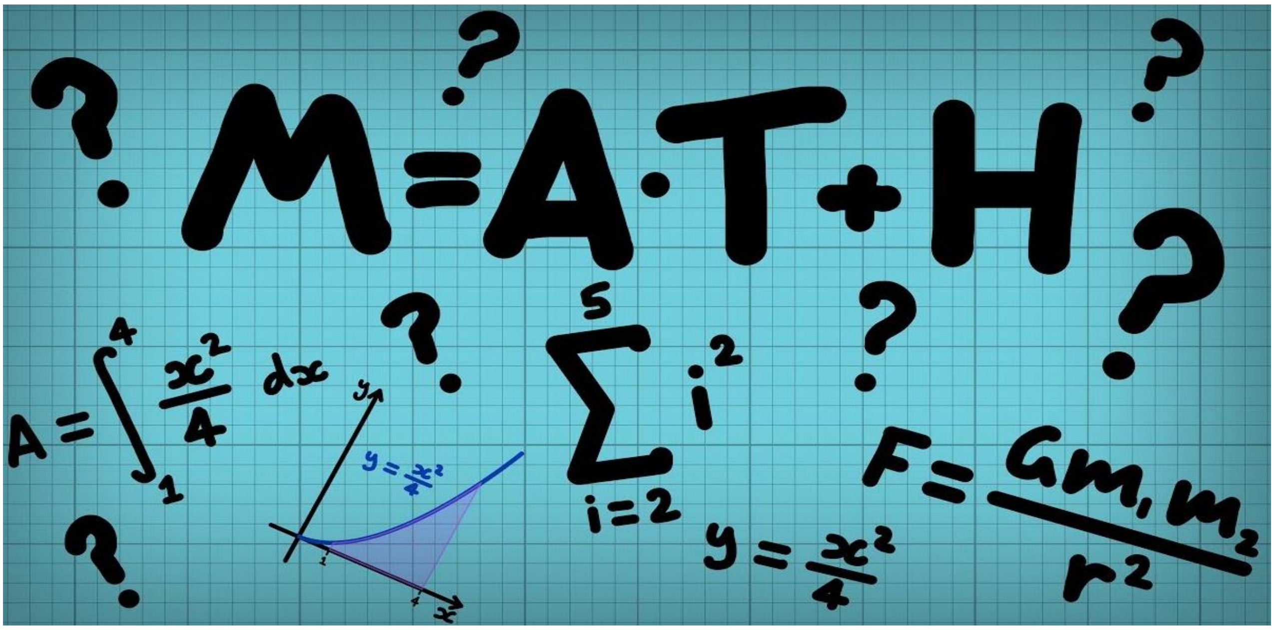
BHADRAK ENGINEERING SCHOOL & TECHNOLOGY (BEST),
ASURALI, BHADRAK

Engineering Mathematics- III (Th: 01)

(As per the 2019-20 syllabus prepared by the SCTE&VT, Bhubaneswar, Odisha)



Third Semester

Common to Electrical & E & TC Engg.

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CHAPTER NO.– 1

COMPLEX NUMBERS

Learning Objective:

1.1 Real and Imaginary numbers.
1.2 Complex numbers, conjugate complex numbers, Modulus and Amplitude of a complex number.
1.3 Geometrical Representation of Complex Numbers.
1.4 Properties of Complex Numbers.
1.5 Determination of three cube roots of unity and their properties
1.6 De Moivre's theorem
1.7 Solve problems on 1·1 - 1·6

1.1:-REAL AND IMAGINARY NUMBERS.

Real numbers are simply the combination of rational and irrational numbers, in the number system. In general, all the arithmetic operations can be performed on these numbers and they can be represented in the number line, also. At the same time, the **imaginary numbers** are the un-real numbers, which cannot be expressed in the number line and is commonly used to represent a **complex number**. Some of the examples of real numbers are 23, -12, 6.99, 5/2, π, and so on.

We have the knowledge of integers, fractions and irrational number (all these constitute real numbers). But if we try to solve the equation $x^2 + 1 = 0$, we observe that these numbers are not adequate. Trying to solve this equation, we arrive at $x^2 = -1$ i.e., $x = \sqrt{-1}$

Square of a positive real number is positive and that of a negative real is also positive. So there is no real number whose square is negative. So we are to create a new kind of number. We define the square root of a negative number as imaginary number' particularly $\sqrt{-1} = i$ the basic imaginary number.

Imaginary numbers:

Taking $i = \sqrt{-1}$ we observe that

i^2=-1, $i^3 = -i$, $i^4 = 1$, $i^5 = i$ so on.

1.2:-COMPLEX NUMBERS

The numbers of the form a + ib where a and b are real numbers and $\sqrt{-1} = i$ are known as complex numbers.
In complex number z = a + ib, the real numbers a and b are respectively known as real and imaginary parts of z and we write:

Re(z) = a and Im (z) = b

Thus the set C of all complex numbers is given by C = {z : z = a + ib, where a, b $\in$ R}

Purely real and purely imaginary numbers:

A complex number z = a + ib is said to be

(i) Purely real, if Im (z) = 0

(ii) Purely imaginary, if Re (z) = 0

Thus 2, -7, $\sqrt{3}$ etc are all purely real numbers

While 2i, $i\sqrt{3}$, $\frac{-1}{2}i$ etc are purely imaginary

Conjugate of a complex number:

The conjugate of a complex number 'z', denoted by $\bar{z}$ is the complex number obtained by changing the sign of imaginary part of z.

Ex. $\overline{2+3i}$=2-3i, $\overline{6i} = -6i$, $\overline{-2i} = 2i$

Modulus of a complex number:

If z = x + iy be a complex number, the modulus of z, written as | z | is a real number $\sqrt{x^2 + y^2}$

For z= 3+4i | z |=$\sqrt{3^2 + 4^2}$ =5

If z=x+iy, $\bar{z} = x - iy$.

| z |=$\sqrt{x^2 + y^2}$,$|\bar{z}| = \sqrt{x^2 + (-y^2)} = \sqrt{x^2 + y^2}$,

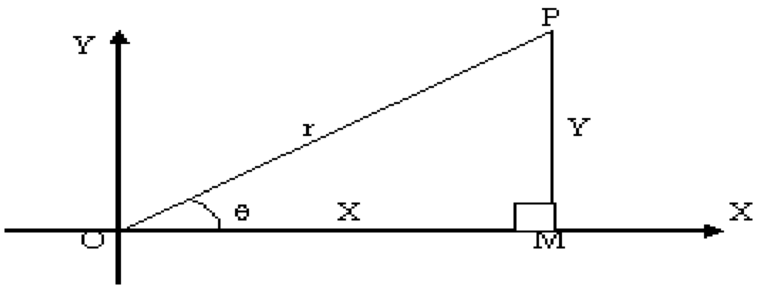
1.3: -Geometrical Representation of a complex number

Let O be the origin and X'OX and YOY' be the co-ordinate axes. The real numbers are taken along X-axis and the imaginary numbers along the Y axis. So the x-axis is called the real axis and the y axis is called the imaginary axis. Then, any complex number z=x+iy may be represented by a unique point p whose co-ordinates are (x, y).
Thus a purely real number will be represented by some point on x-axis.
And, therefore x-axis is called the real axis similarly, purely imaginary numbers lie on y-axis. So, y-axis is called the imaginary-axis.

The form z = r (cosθ + isinθ) is called the polar form or trigonometrical form or standard form or modulus, amplitude form of z.

Here r = $|z|and$ the angle θin known as the amplitude or argument of z written as amp (z) or arg (z)

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) and\ r = \sqrt{x^2 + y^2} = |z| \ (modulus\ of\ z)$$



Argument/Amplitude of a complex number:

The form z = r(cosθ + isinθ) is called the polar form or trigonometrical form or standard form or modulus, amplitude form of z.

Here r = $|z|and$ the angle θin known as the amplitude or argument of z written as amp (z) or arg (z).
Amp(z), θ=$\tan^{-1}\frac{y}{x}$

1.4:-Properties of Complex Numbers.

SUM DIFFERENCE AND PRODUCT OF COMPLEX NUMBERS

For any complex number
z_1 = (a + ib) and z_2 = (c + id)we define

(i) $z_1 + z_2$ = (a + ib) + (c + id) = [(a + c) + i (b + d)]
(ii) $z_1 – z_2$ = (a + ib) – (c + id) = [(a – c) + i (b – d)]
(i) z_1z_2 = (a + ib)(c + id) = [(ac – bd) + i (ad + bc)]

1.5:- **CUBE ROOT OF UNITY**

Let $\sqrt[3]{1} = x, then$

$x^3 = 1$ [on cubing both sides]

$x^3 - 1 = 0$

$(x-1)(x^2+x+1) = 0$

$x - 1=0$ or $x^2 + x + 1 = 0$

$x=1$ or $x = \frac{-1\pm\sqrt{1-4}}{2}$

$x=1$ or or $x = \frac{-1\pm i\sqrt{3}}{2}$

properties

- The cube root of unity are 1, $\frac{-1+i\sqrt{3}}{2}$ and $\frac{-1-i\sqrt{3}}{2}$
- Clearly one of the roots of unity is real and the other two are complex.
- Each complex cube root of unity is the square of the other.

Remarks: If we denote one of the complex cube roots of unity by ω(omega), then clearly the other one is ω^2.

Also, since ω is a cube root of unity, $\omega^3=1$

- The sum of cube roots of unity is zero.

$1+\omega+\omega^2=0$

Example – 1 : **Express in the form a + ib of** $\frac{3+5i}{2-3i}$

Sol^n ***:*** $\frac{3+5i}{2-3i} = \frac{(3+5i)(2+3i)}{(2-3i)(2+3i)}$

$= \frac{6+10i+9i+15i^2}{4-9i^2}$

$= \frac{-9+19i}{13} = \frac{-9}{13} + \frac{19}{13}i$

Example – 2 : **Find the value of** $i^{17} + i^{20} - i^{13}$

Sol^n ***:*** $i^{17} + i^{20} - i^{13} = i^{16} . i + i^{20} - i^{12} . i = (i^2)^8 . i + (i^2)^{10} - (i^2)^6 . i$

$= (-1)^8 i + (-1)^{10} - (-1)^6 i = i + 1 - i = 1$

Example – 3 : **If 1, ω, ω^2 are the cube roots of unity prove that(a) $(1 - \omega)(1 - \omega^2)(1 - \omega^4)(1 - \omega^5) = 9$**

Sol^n***:*** L.H.S. $(1 - \omega)(1 - \omega^2)(1 - \omega^4)(1 - \omega^5)$

$= (1 - \omega)(1 - \omega^2)(1 - \omega^3 . \omega)(1 - \omega^3\omega^2)$

$= (1 - \omega)(1 - \omega^2)(1 - \omega)(1 - \omega^2)$

$= (1 - \omega)^2(1 - \omega^2)^2 = [(1 - \omega)(1 - \omega^2)]^2$

$= [(1 - \omega - \omega^2 + \omega^3]^2 = (2 - \omega - \omega^2)^2$

$= (2 + 1)^2 = 3^2 = 9$

Example – 4 : **Find square roots of**

3 + 4i

Sol^n ***:*** **(a)** Let x, y ∈ R, $x + iy\sqrt{3 + 4i}$

$x^2 - y^2 + i\,2xy = 3 + 4i$

Equating real and imaginary parts$x^2 -$

$y^2 = 3$ and $2xy = 4$

$(x^2 + y^2)^2 = (x^2 - y^2)^2 + 4x^2y^2 = 25$

Hence $x^2 + y^2 = \pm 5$, But since $x^2 + y^2$ is non-negative, we have$x^2 + y^2$

$= 5$

$x^2 - y^2 = 32x^2 = 8$

i.e, $x^2 = 4$, i.e, $x = \pm 2$, $y^2 = 1$ i.e., $y = \pm 1$Hence
square root of $3 + 4i = \pm(2 + i)$

1.6: -De-Movire's Theorem

Theorem: If n is an integer, positive or negative or zero.
Then $(\cos\theta + i\sin\theta)^n = (\cos n\theta + i\sin n\theta)$
Proof: Case –I : When n is a positive integer.
By actual multiplication, we have $(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$
$= (\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)$
$=\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)$
Similarly,
$=(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)(\cos\theta_3 + i\sin\theta_3)$
$=[(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)](\cos\theta_3 + i\sin\theta_3)$
$=[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)](\cos\theta_3 + i\sin\theta_3)$
$=[\cos(\theta_1 + \theta_2 + \theta_3) + i\sin(\theta_1 + \theta_2 + \theta_3)]$
Proceeding in this way, for n factors we have
$(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)\ldots\ldots(\cos\theta_n + i\sin\theta_n)$
$= \cos(\theta_1 + \theta_2 + \ldots.+\theta_n) + i\sin(\theta_1 + \theta_2 + \ldots.+\theta_n)$…..(i)
Putting $\theta_1 = \theta_2 = \theta_3 = \ldots\ldots = \theta$ in (i),
we get
$(\cos\theta + i\sin\theta)^n = (\cos n\theta + i\sin n\theta)$
Case – II : When n is a Negative integer.
Let n = - m, where m is a positive integer.
Then, $(\cos\theta + i\sin\theta)^n = (\cos\theta + i\sin\theta)^{-m} = \frac{1}{(\cos\theta + i\sin\theta)^m}$
$=\frac{1}{(\cos\theta + i\sin\theta)^m} = \frac{1}{\cos m\theta + i\sin m\theta}$ (by case I)
$=\frac{1}{\cos m\theta + i\sin m\theta} \times \frac{(\cos m\theta - i\sin m\theta)}{(\cos m\theta - i\sin m\theta)}$
$= \frac{(\cos m\theta - i\sin m\theta)}{(\cos^2 m\theta - i^2\sin^2 m\theta} = \frac{\cos m\theta - i\sin m\theta}{(\cos^2 m\theta + \sin^2 m\theta}$
$= \cos m\theta - i\sin m\theta = \cos(-m)\theta + i\sin(-m)\theta$
$[\therefore \cos(-\theta) = \cos\theta$ & $\sin(-\theta) = -\sin\theta]$
$= \cos n\theta + i\sin n\theta$ $[\therefore -m = n]$
Case – III : When n=0
Clealy, $(\cos\theta + I\sin\theta)^0 = 1$
$= \cos(0.\theta) + I\sin(0.\theta)$
Thus, De Movire's theorem is true for all integral value of n.
Remark : The theorem is also true when n is fraction
Cor –I: $(\cos\theta + i\sin\theta)^{-n}$
$= \cos(-n)\theta + i\sin(-n)\theta = \cos n\theta - i\sin n\theta$
Cor – 2: $(\cos\theta - i\sin\theta)^n = [\cos(-\theta) + i\sin(-\theta)]^n$
$= \cos[n(-\theta)] + I\sin[n(-\theta)] = \cos(-n\theta) + i\sin(-n\theta)$
$= (\cos n\theta - i\sin n\theta)$
Cor – 3: $(\cos\theta - i\sin\theta)^{-n} = [\cos(-\theta) + \sin(-\theta)]^{-n}$
$=\cos\{(-n)(-\theta)\} + i\sin\{(-n)(-\theta)\} = \cos n\theta + i\sin n\theta$

POSSIBLE LONG QUESTIONS WITH ANSWER

(a) Find the value of $(-i)^{4n+2}$ [2007(w)

Sol^n : $(-i)^{4n+2}$

= (-i)4n (-i)2 = 1 x i2 = -1

(b) Find the modulus & conjugate of 3 – 2i

Sol^n : The modulus of 3-2i

$=\sqrt{3^2+(-2)^2} = \sqrt{13}$

The conjugate of 3-2i = 3+2i

(c) Find the modulus and the argument of (-3 -4i).

Sol^n: ,

The modulas of -3-4i

$= \sqrt{(-3)^2+(-4)^2} = \sqrt{25} = 5$

Argument, θ = tam^{-1} $(\frac{a}{b})$ =tan^{-1} $(\frac{3}{4})$

(h) Express in the form of a+ib [2008(w)

$\frac{(1+i)^2}{3-i}$

Sol^n : $\frac{(1+i)^2}{3-i} = \frac{1+i^2+2i}{3-i} = \frac{1-1+2i}{3-i}$

$= \frac{2i}{3-i} = \frac{2i(3+i)}{(3-i)(3+i)} = \frac{6i+2i^2}{9+1}$

$= \frac{6i-2}{10} = \frac{2(3i-1)}{10} = \frac{-1+3i}{5} = \frac{-1}{5} + \frac{3}{5}i$

(i) If ω be the cube root of unity then find the value of $(1+\omega)^5$.

Sol^n : $(1+\omega)^5 = (-\omega^2)^5 = (-\omega)^{10}$

$= \omega^{10} = (\omega^3)^3 . \omega = 1. \omega = \omega$

(j) Express in the form of A+iB

$\frac{cos\theta - i\sin\theta}{cos\theta + i\sin\theta}$

Sol^n : $\frac{\cos\theta - i\sin\theta}{\cos\theta + i\sin\theta} = \frac{(cos\theta - i\sin\theta)(cos\theta - i\sin\theta)}{(cos\theta + i\sin\theta)(cos\theta - i\sin\theta)}$

$= \frac{cos^2\theta + i^2 sin^2\theta - i(sin\theta\, cos\theta + sin\theta\, cos\theta)}{cos^2\theta + sin^2\theta}$

$= \frac{(cos^2\theta - sin^2\theta) - i\,(2sin\theta cos\theta)}{1}$ =cos 2θ- i sin 2θ

POSSIBLE LONG QUESTIONS

1. If w be the cube roots of unity, then prove that
   $(1 - w + w^2)^7 + (1 + w + w^2)^7 = 128$
2. Find square roots of –5 + 12i [2013(W)
3. $If\ x + \frac{1}{x} = 2\cos\theta \quad then\ show\ that \quad x^n + \frac{1}{x^n} = 2\cos n\theta \quad [2013(w), 2017(w)$
4. $Show\ that\ \ (\frac{1+\sin\theta + i\cos\theta}{1+\sin\theta - i\cos\theta})^n = \cos(\frac{n\pi}{2} - \theta) + i\sin(\frac{n\pi}{2} - \theta) \quad [2008(w)$
5. **If 1, ω, ω^2 are the cube roots of unity prove that**
   $(1+5\omega^2+\omega^4)(1+5\omega+\omega^2)(5+\omega+\omega^2) = 64 \quad [2009(w)]$

CHAPTER – 2
MATRICES

2.1. Define rank of a matrix.
2.2. Perform elementary row transformations to determine the rank of a matrix.
2.3. State Rouche's theorem for consistency of a system of linear equations in unknowns.
2.4. Solve equations in three unknowns testing consistency.

Definition: A matrix is a rectangular array of numbers. A matrix with 'm' rows and 'n' columns is called m×n matrix. A rectangular array of elements of the form

$$A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_n \end{bmatrix}$$

Minor – Minor is the determinate value which is obtained by deleting row & coloumn of theparticular element and denoted by the symbol........................ , i-rows j-coloum.

Ex:-$\begin{bmatrix} 2 & 3 & 5 \\ -2 & 5 & 1 \\ 6 & 0 & 2 \end{bmatrix}$

Minor M_{11}=$\begin{bmatrix} 5 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 2 \end{bmatrix} = 5 \times 2 - 1 \times 0 = 10$

Minor M_{23}=$\begin{bmatrix} 2 & \cdots & 3 \\ \vdots & \ddots & \vdots \\ 6 & \cdots & 0 \end{bmatrix} = 2 \times 0 - 3 \times 6 = -18$

Upper triangular Matrix – A matrix is said to be upper triangular if the elements belowthe main diagoned are zeros.

A=$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 3 \end{bmatrix}$

Sub- Matrix – Any matrix obtained by omitting some rows or columns or both of a given m×n matrix'A' is called a sub-matrix of A

Thus $\begin{bmatrix} a & b \\ d & e \end{bmatrix}$ is a sub-matrix of $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

2.1:Rank of a matrix : A matrix is said to be of rank 'r' if

(i) It has atleast one non-zero minor of order 'r'

(ii) Every minor of order higher than 'r' vanishes.

Otherwise, rank of matrix which can be obtained by eliminating largest order of non vanishing minor of the matrix.

The rank of a matrix A shall be denoted by the symbol r(A).

The rank of a non singular square matrix of order 'n'is n and that of a singular square matrix of order 'n'is less than n.

Obviously $r(A) \leq \min(m,n)$

Remarks:-If all the elements of a matrix A are zero,then r(A)=0 that is the rank of a null matrix is assumed to be zero.

Ex-Find the rank of $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$

Sol^n:-Let A=$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$

r(A))$\leq$ min (3,3)

$$|A| = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix} = 1(15-16) - 2(10-12) + 3(8-9) = -1 + 4 - 3 = 0$$

Here A is a singular square matrix in which there is atleast one 2×2 sub-matrix ,for

$$|A_1| = \begin{vmatrix} 3 & 4 \\ 4 & 5 \end{vmatrix} = 15 - 16 = -1 \neq 0 \; whose\ det. is\ not\ equal\ to\ zero.$$

$Hence\ the\ r(A) = 2$

Ex-Find the rank of $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

Sol^n:-Let A=$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

r(A))$\leq$ min (3,3)

$$|A| = \begin{vmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{vmatrix} = 0 - 0 + 1(0-2) = -2 \neq 0$$

Therefore rank of this matrix is 3.

2.2:**Elementary transformations**: – The following operations three of which refer to rows areknown as elementary transformations.

II. The interchange of any two rows (Rij)

III. The multiplication of any row by a non-zero scalar (kRi)

IV. The addition of a constant multiple of the elements of any row to the corresponding elementsof any other row (Ri + kRj)

Working Rule :

Step – I : Conver the matrix to the upper triangular form.

Step – II : The no.of non-zero rows is the rank of the matrix

Ex-Find the rank of matrix A= $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 4 & 6 & 8 \end{bmatrix}$

Sol^n:- A= $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 4 & 6 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -4 \\ 0 & -2 & -4 \end{bmatrix}$ $[R_2 \to R_2 - 3R_1 \; and \; R_3 \to R_3 - 4R_1]$

$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -4 \\ 0 & 0 & 0 \end{bmatrix}$ $[R_3 \to R_3 - R_2]$

Hence rank of the matrix is r(A)=2

Consistency : A system of equatiars are said to be consistent if either they will have uniquesolution on many solution and sid to be inconsistent if they will have no solution.

$2x + 3y = 8$	$x + 2y = 5$	$x - y = 10$
$x - 2y = 4$	$2x + 4y = 10$	$3x - 3y = 15$
(unique solution)	(many soluion)	(No solution)

Consistency of a system of linear equations : -

Consider a system of m linear equations

$a_{11}x_1 + a_{12}x_2 + ... + a_{1x}x_n = b_1$

$a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2$

... . -

... .

$a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n = b_m$

Containing the n unknows $x_1, x_{2........}x_n$.

Considering set of equations

$a_1x + b_1y + c_1z = d_1$

$a_2x + b_2y + c_2z = d_2$

$a_3x + b_3y + c_3z = d_3$

The equation can be written in matrix form as

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

AX=D, where A is the co-efficient matrix

Now we join matrix A and D

$$[A:D] = \begin{bmatrix} a_1 & b_1 & c_1 & : & d_1 \\ a_2 & b_2 & c_2 & : & d_2 \\ a_3 & b_3 & c_3 & : & d_3 \end{bmatrix}$$

It is called Augumented matrix

2.3:Rouche's Theorem :

The system of equations is consistent if and only if the co-efficient matrix A and the augumented matrix K are of some rank otherwise the system is inconsistent.

Procedure to test the consistency of a system of equations in x unknows.

Find the ranks of the co-efficient matrix A and the augmented matrix 'K' by reducing tothe upper triangular form by elementary row operations.

(a) Consistant equations : If Rank A = Rank K

(i) Unique solution Rank A = Rank K = n

Where n = number of unknowns.

(ii) Infinite solution : Rank A = Rank K = r. r < n.

(b) Inconstant equations if Rank A ≠ Rank K

Example:-Solve

$x + 2y - z = 3$

$3x - y + 2z = 1$

$2x - 2y + 3z = 2$

Solution :

Writing the above equations in matrix form

AX=D

$$\begin{bmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 2 & -2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

The given equation can be represented in Augumented matrix form

$$K=\begin{bmatrix} 1 & 2 & -1 & 3 \\ 3 & -1 & 2 & \vdots\, 1 \\ 2 & -2 & 3 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & \vdots\, -8 \\ 0 & -4 & 5 & -4 \end{bmatrix} \begin{pmatrix} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 2R_1 \end{pmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 5 & -7 & \vdots\, -8 \\ 0 & 5 & -4 & -4 \end{bmatrix} (c_2 \leftrightarrow c_3)$$

$$\sim \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 5 & -7 & \vdots\, -8 \\ 0 & 0 & 3 & 4 \end{bmatrix} (R_3 \to R_3 - R_2)$$

r(A)=3 and r(K)=3

r(A)= r(K) →equations are consistent.

Next to find its solution

$x - y + 2z = 3$

$5y - 7z = -8$

$3z = 4$

By backward substitution we get z = 4/3

$5y - 7z = -8$, $5y = -8 + 7z$, $y = \frac{-8+7\times4/3}{5} = \frac{4}{15}$

x –y+2z = 3 , x = 3+y–2z, $x = 3 + \frac{4}{15} - 2 \times \frac{4}{3} = \frac{9}{15}$

$x = \frac{9}{15}, y = \frac{4}{15}$ *and* $z = \frac{4}{3}$ *be the required solution.*

Example:-For what value of α and β do the system of equations

x + y+z = 6

x +2y+ 3z =10

x +2y+ αz = β have

(i) No solution,(ii) unique solution,(iii) infinite solutions.

Solution :

Writing the above equations in matrix form

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & A \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ \beta \end{bmatrix}$$

The given equation can be represented in Augumented matrix form

$$K = \begin{bmatrix} 1 & 1 & 1 & \vdots & 6 \\ 1 & 2 & 3 & \vdots & 10 \\ 1 & 2 & \alpha & & \beta \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & & 6 \\ 0 & 1 & 2 & \vdots & 4 \\ 0 & 1 & \alpha - 1 & & \beta - 6 \end{bmatrix} \begin{pmatrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{pmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & & 6 \\ 0 & 1 & 2 & \vdots & 4 \\ 0 & 0 & \alpha - 3 & & \beta - 10 \end{bmatrix} (R_3 \to R_3 - R_2)$$

Case-I : If $\alpha = 3, \beta \neq 10$

Then the rank of coefficient matrix r(A)=2

And rank of augumented matrix, r(K)=3

So, r(A) ≠ r(K), the system has no solution.

Case-II : If $\alpha \neq 3, \beta$ may have any value

Then the rank of coefficient matrix r(A)=3

And rank of augumented matrix, r(K)=3

So, r(A)= r(K)= no.of unknowns, the system has unique solution.

Case-I : If $\alpha = 3, \beta = 10$

Then the rank of coefficient matrix r(A)=2

And rank of augumented matrix, r(K)=2

So, r(A)= r(K)=2< no.of unknowns,

the system has an infinite number of solution.

SYSTEM OF LINEAR HOMOGENEOUS EQUATIONS

Consider a system of m linear equations

$a_{11}x_1 + a_{12}x_2 + ... + a_{1x}x_n = 0$

$a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = 0$

... . -

... .

$a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n = 0$

Containing the n unknows $x_1, x_{2........}x_n$.

Find the rank ‘r’ of the coefficient matrix A by reducing it to the triangular form by elementary row operations.

(a) If r = n→the equation have only trival solution i.e $x_1=x_2=x_3=\ldots\ldots\ldots\ldots x_n=0$

(b) If r < n → the equation an infinite number of non-trival solutions.

To obtain infinite solution,set(n-r) variables any arbitrary value and solve for the remaining unknowns.

Example:-Solve the following system of equations.

$x–2y+z =0$

$x+2y–z =0$

$2x +y+ 3z =0$

Solution :

Writing the equations in the form AX=0, we have

$$\begin{bmatrix}1 & -1 & 1\\ 1 & 2 & -1\\ 2 & 1 & 3\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$Where\ \mathrm{A}=\begin{bmatrix}1 & -1 & 1\\ 1 & 2 & -1\\ 2 & 1 & 3\end{bmatrix}$$

$$A \sim \begin{bmatrix}1 & -1 & 1\\ 0 & 3 & -2\\ 0 & 3 & 1\end{bmatrix}\begin{pmatrix}R_2 \to R_2 - R_1\\ R_3 \to R_3 - 2R_1\end{pmatrix}$$

$$A \sim \begin{bmatrix}1 & -1 & 1\\ 0 & 3 & -2\\ 0 & 0 & 3\end{bmatrix}(R_3 \to R_3 - R_2)$$

$$A \sim \begin{bmatrix}1 & -1 & 1\\ 0 & 1 & -\frac{2}{3}\\ 0 & 0 & 1\end{bmatrix}\begin{pmatrix}R_2 \to R_2 \times \frac{1}{3}\\ R_3 \to R_3 \times \frac{1}{3}\end{pmatrix}$$

r(A)=3=number of variables and hence the given equations have only trival solutions.

x=y=z=0

Example:-Solve the following system of equations.

x+2y+3z =0

$2x$+3y+4z =0

7x +13y+ 19z =0

Solution :

Writing the equations in the form AX=0, we have

$$\begin{bmatrix}1 & 2 & 3\\2 & 3 & 4\\7 & 13 & 19\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$Where$ A=$\begin{bmatrix}1 & -1 & 1\\1 & 2 & -1\\2 & 1 & 3\end{bmatrix}$

$$A \sim \begin{bmatrix}1 & 2 & 3\\0 & -1 & -2\\0 & -1 & -2\end{bmatrix}\begin{pmatrix}R_2 \to R_2 - R_1\\R_3 \to R_3 - 2R_1\end{pmatrix}$$

$$A \sim \begin{bmatrix}1 & 2 & 3\\0 & -1 & -2\\0 & 0 & 0\end{bmatrix}(R_2 \to R_2 - R_3)$$

$The\ rank\ of\ matrix\ A\ is\ 2 < number\ of\ variables$

Hence the system has infinitely many solutions.

Now omitting (m-r) i.e,omitting third equation and writing first two as follows(containing two unknowns x,y and taking z=constant) we have

$x + 2y + 3z = 0$

$-y - 2z = 0$

i.e y=$-2z(let\ z = k)$

y=$-2k$,

$x - 4k + 3k = 0$

$x = k$

Hence the general solution is

$x = k$, y=$-2k$, $z = k$

SHORT QUESTIONS WITH ANSWER

Q.1 Determine the rank of the matrix A=$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$ [2008(W)

Sol^n: Let$|A| = 1, \neq 0$

i.e Rank of A r(A)=3

Q.2 Determine the rank of the matrix A=$\begin{bmatrix}1 & 2 & 3\\2 & 3 & 4\\3 & 4 & 5\end{bmatrix}$

Sol^n: Let A=$\begin{bmatrix}1 & 2 & 3\\2 & 3 & 4\\3 & 4 & 5\end{bmatrix}$

r(A)≤ min(3,3)

$|A| = 1(15 - 16) - 2(10 - 12) + 3(8 - 9) = -1 + 4 - 3 = 0$

Here A is asingular square matrix in which there is atleast one 2×2 sub-matrix

$|A_1| = \begin{vmatrix} 3 & 4 \\ 4 & 5 \end{vmatrix} = 15 - 16 = -1 \neq 0$

Hence the rank of A=2.

Q.3 Define rank of a matrix. [2005(W)

Sol^n:The rank of a matrix is the largest order of the non-vanishing minor of that matrix.

Q.4 Determine the rank of the matrix A=$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 4 & 8 & 0 \end{bmatrix}$ [2009(W)

Sol^n: Let$|A| = 0$,

i.e Rank of A r(A)=1

LONG QUESTIONS

Q.1 Check whether the following system of equations is consistent or not.solve if it is consistent.

$4x + 3y + 2z = -7$

$2x + y - 4z = -1$

$x + 2y + z = 1$

Q.2 Test the consistency & solve

$4x - 5y + z = 2$

$3x + y - 2z = 9$

$x + 4y + z = 5$

Q.3 For what value of λ and μ the equations [2008,2005]

$x + 2y + z = 8$

$2x + y + 3z = 13$

$3x + 4y - \lambda z = \mu \text{ } have$

(i) No solution (ii) unique solution (iii) infinite solutions.

Q.4 Solve the following system of equations [2009]

$x - y + 2z - 3w = 0$

$3x + 2y - 4z + w = 0$

$4x - 2y + 9w = 0$