BHADRAK ENGINEERING SCHOOL & TECHNOLOGY
(BEST), ASURALI, BHADRAK

Strength of Material

(Th- 02)

(As per the 2020-21 syllabus of the SCTE&VT, Bhubaneswar, Odisha)

Third Semester

Mechanical Engg.

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STRENGTH OF MATERIAL

TOPIC WISE DISTRIBUTION PERIODS

Sl. No.	Name of the chapter as per the syllabus	No of Periods as Per the Syllabus	No of Periods Actually Needed	Expected marks
01	Simple stress and strain	10	13	22
02	Thin cylinder and spherical shell under internal pressure	08	07	09
03	Two-dimensional stress system	10	09	17
04	Bending moment and shear force	10	12	19
05	Theory of simple bending	10	07	19
06	Combined direct and bending stress	06	06	07
07	Torsion	06	06	17
	TOTAL	**60**	**60**	**110**

CHAPTER NO. – 01

SIMPLE STRESS AND STRAIN

LEARNING OBJECTIVES:

1.1 Types of loads, stresses & strains, (Axial and tangential) Hooke's law, young's modulus, bulk modulus, modulus of rigidity, Poisson's ratio, derive the relation between three elastic constants,
1.2 Principle of super position, stresses in composite section
1.3 Temperature stress, determine the temperature stress in composite bar (single core)
1.4 Strain energy and resilience, Stress due to gradually applied, suddenly applied and impact load
1.5 Simple problems on above.

INTRODUCTION:

- The subject strength of materials is basically a study of
  I. The behaviour of materials under various types of loads and moments.
  II. The action of forces and their effects on structural and machine elements such as angle irons, circular bars and beams etc.
- The knowledge thus acquired provides rational approach to all design problems, i.e., it helps an engineer to design all types of machines and structures and suggest protective measures for the safe working conditions of such elements. The different structural components may be:
  I. Trusses, beams and columns of buildings and bridges
  II. Power transmission shafts, springs and pressure vessels
  III. Mechanical components in the aircraft, and in the electrical/electronic products

1.1 Types Of Load, Stresses & Strains, (Axial And Tangential) Hooke's Law, Young's Modulus, Bulk Modulus, Modulus Of Rigidity, Poisson's Ratio, Derive The Relation Between Three Elastic Constants.

Load:

- A load may be defined as the combined effect of external forces acting on a body.

Classification of Loads:

- The loads may be classified as:
  i. Dead loads
  ii. Live or fluctuating loads
  iii. Inertia loads or forces
  iv. Centrifugal loads or forces

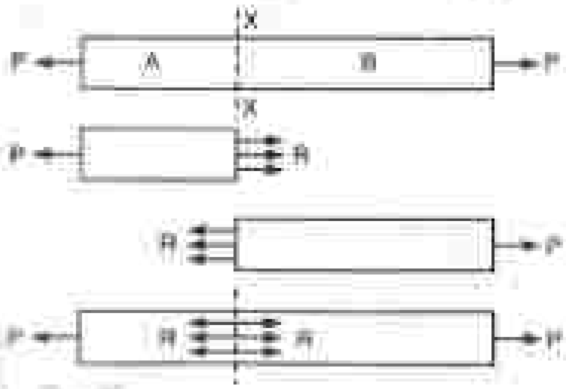
- The other way of classification is
    i. Tensile loads
    ii. Compressive loads
    iii. Torsional or twisting loads
    iv. Bending loads
    v. Shearing loads
- The load may also be a **point (or concentrated)** or **distributed**.

Stress:

- The internal resistance per unit area offered by the material of the body against external loading is called intensity of stress or simply called as stress.

Or

- The internal resistance which the body offers to meet with the load is called stress.
- It is denoted by the symbol 'σ' called sigma.



- Mathematically,

$$\sigma = \frac{R}{A} = \frac{P}{A}$$

Where, σ = stress

R = internal resisting force

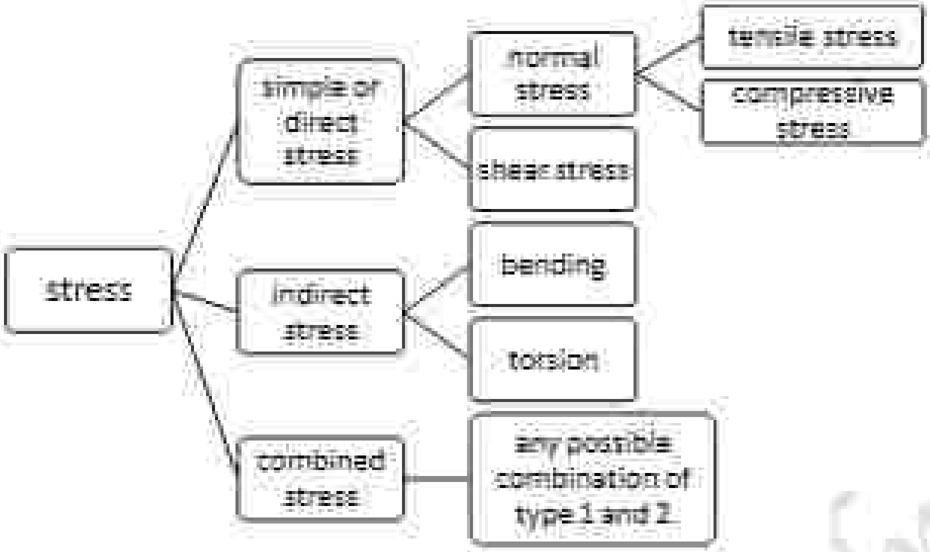
P = Load or external force causing stress to develop

A = area over which stress develops

- Its unit is N/m^2, N/mm^2, KPa, MPa, GPa.
- $1N/m^2$ = 1Pascal, 1Kpa = 10^3 Pascal, 1MPa = 10^6 Pascal, 1GPa = 10^9 Pascal

Classification of stress:

- The various types of stresses may be classified as:



Normal stress:

- The stresses acting normal to the plane on which the forces act are called direct or normal stress.
- The normal stresses are of two types:
  - i. Tensile stress
  - ii. Compressive stress

Tensile Stress (σ_t):

- When a section is subjected to two equal and opposite axial pulls and the body tends to increase its length, then the stress induced is called tensile stress.

- Mathematically,

$$\sigma_t = \frac{P}{A}$$

Compressive stress (σ_c):

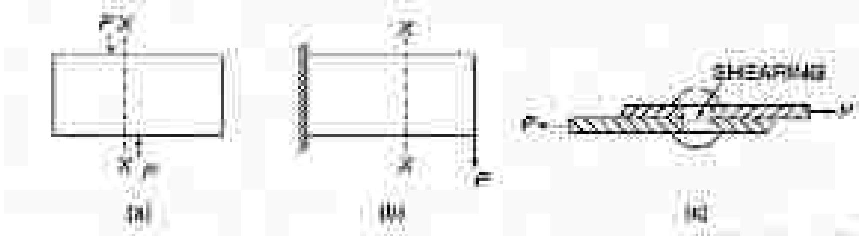
- When a section is subjected to two equal and opposite axial pushes and the body tends to shorten its length, then the stress induced is called compressive stress.

- Mathematically,

$$\sigma_c = \frac{P}{A}$$

Shear Stress:

- When two equal and opposite parallel forces not in the same line act on two parts of a body, then one part tends to slide over or shear from the other across any section and the stress developed is termed as shear stress.
- Shear stress is always tangential to the area over which it acts.
- It is denoted by the symbol 'τ' called 'tau'.



- If P is the force applied and A is the area being sheared, then the intensity of shear stress is given by,

$$\tau = \frac{P}{A}$$

Strain:

- The strain is the deformation produced by stress.
- The ratio of change in dimension to original dimension of a body is called as strain.
- It is denoted by the letter 'e' or 'ε'.
- It is a unit less quantity.
- Strain(e) = $\frac{\text{change in dimension}}{\text{original dimension}}$

Tensile strain (e_t):

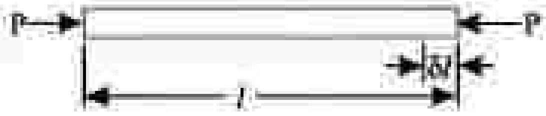
- A piece of material, with uniform cross section, subjected to a uniform axial tensile stress, will increase its length from l to $(l + \delta l)$ and the increase of length δl is the actual deformation of the material.
- It is the ratio of increase in length to the original length of a body.

- Mathematically,

$$e_t = \frac{\delta l}{l}$$

Compressive strain (e_c):

- Under compressive forces, a similar piece of material would be reduced in length from l to $(l-\delta l)$.
- It is the ratio of decrease in length to the original length of a body.



- Mathematically,

$$e_c = \frac{\delta l}{l}$$

Shear strain:

- In case of a shearing load, a shear strain will be produced; this is measured by the angle through which the body distorts.

- Consider a rectangular block LMNP fixed at one face and subjected to force 'F'. After application of force, it distorts through an angle 'ϕ' and occupies new position LM'N'P. the shear strain (e_s) is given by,

$$e_s = \frac{NN'}{NP} = \tan \emptyset$$

$$= \emptyset\ (radians) \ldots \ldots since\ \emptyset\ is\ very\ small$$

- The above result has been obtained by assuming NN' equal to arc (as NN' is small) drawn with centre P and radius PN.

Volumetric strain:

- The ratio between change in volume and original volume of the body is called volumetric strain.
- It is denoted by e_v
- Mathematically,

$$e_v = \frac{change\ in\ volume}{original\ volume} = \frac{\delta v}{v}$$

Elasticity:

- Whenever a body is acted upon by external load, it undergoes some deformation.

- The property by virtue of which the body regains its original shape and size after removal of the external load is called elasticity.

Elastic material:

- If the material regains its original shape and size after removal of the external load, then the material is known as elastic material.

Elastic limit:

- There is always a limiting value of load up to which the strain totally disappears on the removal of load. The stress corresponding to this load is called elastic limit.

Hooke's Law:

- Hooke's law states that when a material is loaded within elastic limit, stress is directly proportional to strain.
- Mathematically,

$$stress \propto strain$$

$$\Rightarrow \frac{stress}{strain} = constant\ (E)$$

- Where the constant of proportionality *E* is called ***Young's modulus or modulus of elasticity***.

Young's Modulus:

- It is defined as, "The ratio of stress to strain".
- It is denoted by the letter '*E*'.
- Mathematically,

$$E = \frac{stress}{strain} = \frac{\sigma}{\varepsilon}$$

- Its unit is same as stress i.e. N/m^2, N/mm^2, KPa, MPa, GPa.

Modulus of Rigidity:

- It is defined as, "The ratio of shear stress to shear stain".
- It is denoted by letter ***C, N or G***.
- Mathematically,

$$C = \frac{shear\ stress}{shear\ strain} = \frac{\tau}{\varepsilon_s}$$

- Its unit is N/m^2, N/mm^2, KPa, MPa, GPa.

Bulk Modulus or volume modulus of elasticity:

- When a body is subjected to three mutually perpendicular stresses, of equal intensity, the ratio of direct stress to the corresponding volumetric strain is known as bulk modules.
- It is denoted by '*K*'
- Mathematically,

$$K = \frac{direct\ stress}{volumetric\ strain} = \frac{\sigma}{e_v}$$

- Its unit is N/m^2, N/mm^2, KPa, MPa, GPa.

Poisson's Ratio:

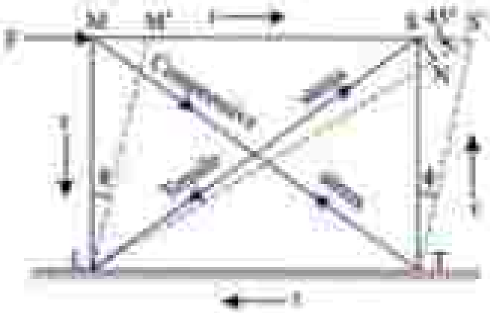
- If a body is stressed within its elastic limit, the lateral strain bears a constant ratio to the linear strain this constant is known as Poisson's ratio.
- It is denoted by symbol 'μ' or 1/m.
- It is unit less.
- Mathematically,

$$\mu = \frac{lateral\ strain}{linear\ strain} = \frac{1}{m}$$

The value of 'μ' lies from 0.25 to 0.34 for different materials.

- Linear strain $= \frac{\sigma}{E}$ and Lateral strain $= \frac{\sigma}{mE}$

Relation between E and C:



Consider a solid cube *LMST* subjected to a shearing force '*F*'.
Due to shearing load '*F*' let the cube is distorted to *LM'S'T* and the diagonal *LS* to *LS'*.
Let τ be the shear stress produced in the faces *MS* and *LT* due to this shearing force.

$$shear\ strain\ (\emptyset) = \frac{SS'}{ST}$$

$$also, shear\ strain = \frac{\tau}{C}$$

$$\therefore \frac{SS'}{ST} = \frac{\tau}{C} \ldots\ldots\ldots\ldots\ldots\ldots (i)$$

On the diagonal *LS'*, draw a perpendicular *SN* from *S*.

$$Now\ diagonal\ strain = \frac{NS'}{LN} = \frac{NS'}{LS} \ldots\ldots\ldots (ii)$$

$$NS' = SS' \cos 45^0 = \frac{SS'}{\sqrt{2}}$$

[$\angle LS'T$ is assumed to be equal to $\angle LST$ since SS' is very small.]

$LS = ST \times \sqrt{2}$

Putting the value of LS in equation (ii), we get

$$Diagonal\ strain = \frac{SS'}{\sqrt{2}\,ST \times \sqrt{2}} = \frac{SS'}{2ST}$$

$$But, \quad \frac{SS'}{ST} = \frac{\tau}{C}$$

$$\therefore Diagonal\ strain = \frac{\tau}{2C} = \frac{\sigma}{2C} \ldots\ldots\ldots\ldots (iii)$$

Where, σ is the normal stress due to shear stress (τ).

$$The\ net\ strain\ in\ the\ direction\ of\ diagonal\ LS = \frac{\sigma}{E} + \frac{\sigma}{mE} = \frac{\sigma}{E}\left[1 + \frac{1}{m}\right] \ldots\ldots\ldots\ldots\ldots (iv)$$

[Since the diagonal LS and MT have normal tensile and compressive stress (σ), respectively.]

Comparing equation (iii) and (iv), we get

$$\frac{\sigma}{2C} = \frac{\sigma}{E}\left[1 + \frac{1}{m}\right]$$

$$\therefore E = 2C\left[1 + \frac{1}{m}\right] \ldots\ldots\ldots (v)$$

Relation Between E and K:

If the solid cube is subjected to σ (normal compressive stress) on all the face,

$$The\ direct\ strain\ in\ each\ axis = \frac{\sigma}{E}\,(compressive)\ and$$

$$lateral\ strain\ in\ other\ axis = \frac{\sigma}{mE}\,(tensile)$$

$$\therefore Net\ compressive\ strain\ in\ each\ axis = \frac{\sigma}{E} - \frac{\sigma}{mE} - \frac{\sigma}{mE} = \frac{\sigma}{E}\left[1 - \frac{2}{m}\right]$$

Volumetric strain (e_v) in each case will be,

$$e_v = 3 \times linear\ strain = 3 \times \frac{\sigma}{E}\left[1 - \frac{2}{m}\right]$$

$$But, \quad e_v = \frac{\sigma}{K}$$

$$\therefore \frac{\sigma}{K} = \frac{3\sigma}{E}\left[1 - \frac{2}{m}\right] \quad or \quad E = 3K\left[1 - \frac{2}{m}\right] \ldots\ldots\ldots\ldots\ldots (vi)$$

Relation between E, C & K:

From equation (v) we get

$$m = \frac{2C}{E - 2C}$$

Substituting the value of m in equation (vi) we have,

$$E = 3K\left[1 - \frac{2}{m}\right]$$

$$\Rightarrow E = 3K\left[1 - \frac{2}{2C/E - 2C}\right]$$

$$\Rightarrow E = 3K\left[1 - \frac{E - 2C}{C}\right]$$

$$\Rightarrow \frac{E}{3K} = \left[\frac{C - E + 2C}{C}\right]$$

$$\Rightarrow \frac{E}{3K} = \left[\frac{3C - E}{C}\right]$$

$$\Rightarrow \frac{E}{3K} = \left[\frac{3C}{C} - \frac{E}{C}\right]$$

$$\Rightarrow \frac{E}{3K} + \frac{E}{C} = 3$$

$$\Rightarrow \frac{EC + 3KE}{3KC} = 3$$

$$\Rightarrow EC + 3KE = 9KC$$

$$\Rightarrow E(3K + C) = 9KC$$

$$\Rightarrow E = \frac{9KC}{3K + C}$$

1.2 Principle of super position, stresses in composite section:

Deformation of a body due to force acting on it:

Consider a body subjected to a tensile stress.

Let, P = Load or force acting on the body,

l = Length of the body,

A = Cross-sectional area of the body,

σ = Stress induced in the body,

E = Modulus of elasticity for the material of the body,

ε = Strain, and

δl = Deformation of the body

We know that

$$\sigma = \frac{P}{A} \qquad and \qquad \varepsilon = \frac{\sigma}{E} = \frac{P}{AE}$$

We also know that,

$$\varepsilon = \frac{\delta l}{l} \Rightarrow \delta l = \varepsilon \times l$$

$$\Rightarrow \delta l = \frac{Pl}{AE}$$

Notes:

1. The above formula holds good for compressive stress also.

2. For most of the structural materials, the modulus of elasticity for compression is the same as that for tension.

3. Sometimes in calculations, the tensile stress and tensile strain are taken as positive, whereas compressive stress and compressive strain as negative.

<u>Example 1</u>: A steel rod 1 m long and 20 mm × 20 mm in cross-section is subjected to a tensile force of 40 kN. Determine the elongation of the rod, if modulus of elasticity for the rod material is 200 GPa.

Solution. Given: Length (l) = 1 m = 1 × 10^3 mm; Cross-sectional area (A) = 20 × 20 = 400 mm^2; Tensile force (P) = 40 kN = 40 × 10^3 N and modulus of elasticity (E) = 200 GPa = 200 × 10^3 N/mm^2.

We know that elongation of the road,

$$\delta l = \frac{P.l}{A.E} = \frac{(40 \times 10^3) \times (1 \times 10^3)}{400 \times (20 \times 10^3)} = \mathbf{0.5\ mm}$$

<u>Example 2</u>: A hollow cylinder 2 m long has an outside diameter of 50 mm and inside diameter of 30 mm. If the cylinder is carrying a load of 25 kN, find the stress in the cylinder. Also find the deformation of the cylinder, if the value of modulus of elasticity for the cylinder material is 100 GPa.

Solution. Given: Length (l) = 2 m = 2 × 10^3 mm; Outside diameter (D) = 50 mm; Inside diameter (d) = 30 mm; Load (P) = 25 kN = 25 × 10^3 N and modulus of elasticity (E) = 100 GPa = 100 × 10^3 N/mm^2.

Stress in the cylinder

We know that cross-sectional area of the hollow cylinder,

$$A = \frac{\pi}{4} \times (D^2 - d^2) = \frac{\pi}{4} \times [(50)^2 - (30)^2] = 1257\ mm^2$$

and stress in the cylinder,

$$\sigma = \frac{P}{A} = \frac{25 \times 10^3}{1257} = 19.9\ N/mm^2 = \mathbf{19.9\ MPa}$$

Deformation of the cylinder

We also know that deformation of the cylinder,

$$\delta l = \frac{P.l}{A.E} = \frac{(25 \times 10^3) \times (2 \times 10^3)}{1257 \times (100 \times 10^3)} = \mathbf{0.4\ mm}$$

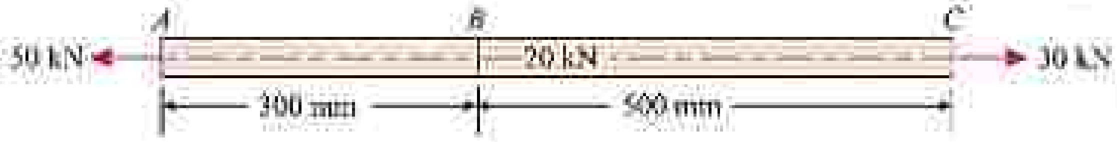
<u>Principle of superposition</u>:

- Sometimes a body is subjected to a number of forces acting on its outer edges as well as at some other sections, at different position along the length of the body. In such a case, the forces are split up and their effects are considered on individual sections. The resulting deformation, of the body, is equal to the algebraic sum of the deformations of the individual sections. This is the principle of superposition which may be stated as:
- ***"The resultant elongation due to several loads acting on a body is the algebraic sum of the elongations caused by individual loads".***
- Mathematically,

$$\delta l = \frac{P_1 l_1}{AE} + \frac{P_2 l_2}{AE} + \frac{P_3 l_3}{AE} \dots\dots\dots$$

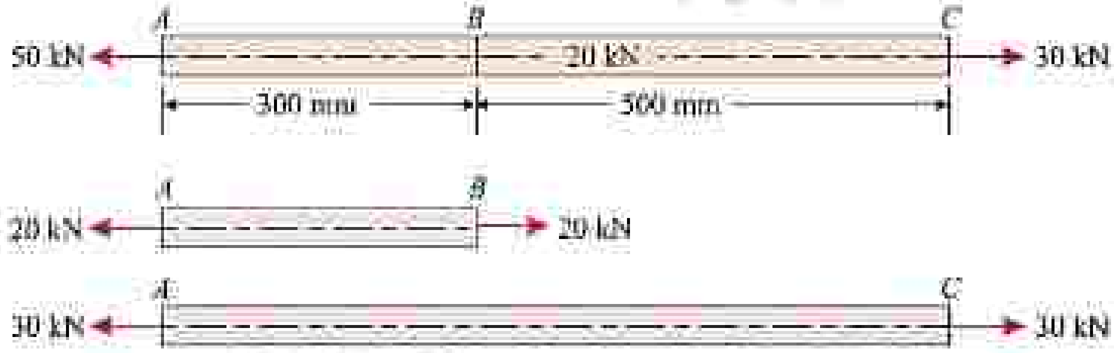
$$= \frac{1}{AE}(P_1 l_1 + P_2 l_2 + P_3 l_3 \dots\dots\dots\dots\dots)$$

***Example 3**: A steel bar of cross-sectional area 200 mm² is loaded as shown in Fig. Find the change in length of the bar. Take E as 200 GPa.*



Solution. Given: Cross-sectional area (A) = 200 mm² and modulus of elasticity (E) = 200 GPa = 200 × 10³ N/mm².

For the sake of simplification, the force of 50 kN acting at A may be split up into two forces of 20 kN and 30 kN respectively. Now it will be seen that part AB of the bar is subjected to a tension of 20 kN and AC is subjected to a tension of 30 kN as shown in Figure below.



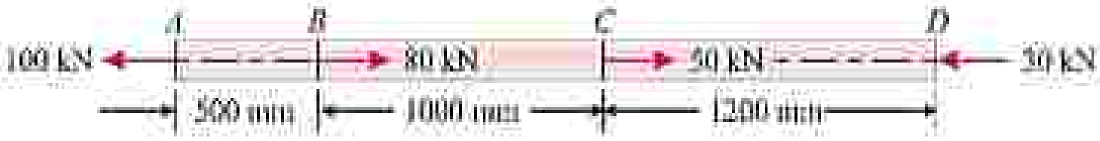
We know that change in length of the bar,

$$\delta l = \frac{1}{AE}(P_1 l_1 + P_2 l_2)$$

$$\Rightarrow \delta l = \frac{1}{200 \times 200 \times 10^3}[[(20 \times 10^3) \times (300)] + [(30 \times 10^3) \times (800)]]$$

$$\Rightarrow \delta l = 0.75\ mm$$

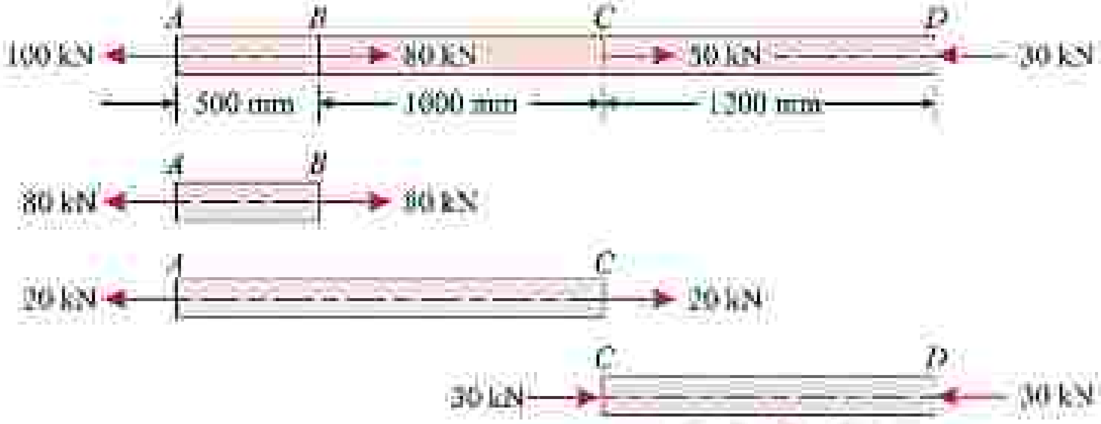
***Example 4**: A brass bar, having cross-sectional area of 500 mm² is subjected to axial forces as shown in Fig. Find the total elongation of the bar. Take E = 80 GPa.*



Solution. Given: Cross-sectional area (A) = 500 mm² and modulus of elasticity (E) = 80 GPa = 80 kN/mm².

For the sake of simplification, the force of 100 kN acting at A may be split up into two forces of 80 kN and 20 kN respectively. Similarly, the force of 50 kN acting at C may also be split up into two forces of 20 kN and 30 kN respectively.

Now it will be seen that the part AB of the bar is subjected to a tensile force of 80 kN, part AC is subjected to a tensile force of 20 kN and the part CD is subjected to a compression force of 30 kN as shown in Figure.



We know that elongation of the bar,

$$\delta l = \frac{1}{AE}(P_1 l_1 + P_2 l_2 + P_3 l_3)$$

$$\Rightarrow \delta l = \frac{1}{500 \times 80}[(80 \times 500) + (20 \times 1500) - (30 \times 1200)]$$

$$\Rightarrow \boldsymbol{\delta l = 0.85\ mm}$$

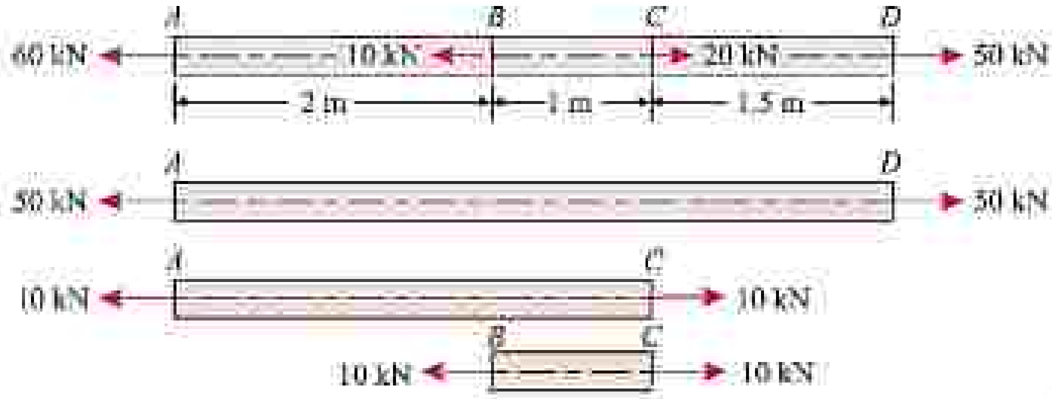
<u>Example 5</u>: A steel rod ABCD 4.5 m long and 25 mm in diameter is subjected to the forces as shown in Fig. If the value of Young's modulus for the steel is 200 GPa, determine its deformation.

Solution. Given: Diameter (D) = 25 mm and Young's modulus (E) = 200 GPa = 200 kN/mm^2.

We know that cross-sectional area of the steel rod,

$$A = \frac{\pi}{4}(D)^2 = \frac{\pi}{4} \times (25)^2 = 491\ mm^2$$

For the sake of simplification, the force of 60 kN acting at A may be split up into two forces of 50 kN and 10 kN respectively. Similarly, the force of 20 kN acting at C may also be split up into two forces of 10 kN and 10 kN respectively.



Now it will be seen that the bar AD is subjected a tensile force of 50 kN, part AC is subjected to a tensile force of 10 kN and the part BC is subjected to a tensile force of 10 kN as shown in Figure above.

We know that deformation of the bar,

$$\delta l = \frac{1}{AE}(P_1 l_1 + P_2 l_2 + P_3 l_3)$$

$$\Rightarrow \delta l = \frac{1}{491 \times 200}[(50 \times 4.5 \times 10^3) + (10 \times 3 \times 10^3) + (10 \times 1 \times 10^3)]$$

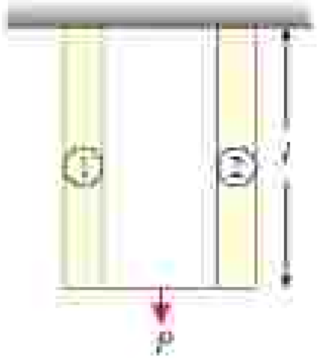
$$\Rightarrow \delta l = \frac{1}{491 \times 200} \times (265 \times 10^3) = \mathbf{2.70\ mm}$$

Stresses in composite section:

A bar made up of two or more different materials, joined together is called a composite bar. The bars are joined in such a manner, that the system extends or contracts as one unit, equally, when subjected to tension or compression. Following two points should always be kept in view, while solving example on composite bars:

1. Extension or contraction of the bar is equal. Therefore strain (i.e., deformation per unit length) is also equal.
2. The total external load, on the bar, is equal to the sum of the loads carried by the different materials.

Consider a composite bar subjected to load P fixed at the top as shown in figure.



Total load is shared by the two bars, such as:

$$P = P_1 + P_2 = \sigma_1 A_1 + \sigma_2 A_2$$

Further elongations in two bars are same, i.e. strains in the bars are equal. Thus

$$\varepsilon_1 = \varepsilon_2 \Rightarrow \frac{\sigma_1}{E_1} = \frac{\sigma_2}{E_2}$$

$$\therefore \frac{\sigma_1}{\sigma_2} = \frac{E_1}{E_2}$$

The ratio E_1/E_2 is called the modular ratio.

Example 6: *A reinforced concrete circular section 50,000 mm² cross-sectional area carrying 6 reinforcing bars whose total area is 500mm². Find the safe load, the column can carry, if the concrete is not to be stressed more than 3.5Mpa. Take modular ratio for steel and concrete as 18.*

Data given:

Area of column (A) = 50,000 mm², No of reinforcing bars = 6, Total area of steel bars (A_s) = 500 mm², Maximum stress in concrete (σ_c) = 3.5 MPa = 3.5 N/mm², Modular ratio (E_s/E_c) = 18.

We know that area of concrete,

$$A_c = 50000 - 500 = 49500\ mm^2$$

And stress in steel,

$$\sigma_s = \frac{E_s}{E_c} \times \sigma_c = 18 \times 3.5 = 63\ N/mm^2$$

∴ Safe load, the column can carry,

$$F = (\sigma_s . A_s) + (\sigma_c . A_c) = (63 \times 500) + (3.5 \times 49500) N$$
$$= 204750\ N = \mathbf{204.75\ kN}$$

Example 7: *A reinforced concrete column 500mm x 500mm in cross section is reinforced with 4 steel bars of 25 mm diameter, one in each corner. The column is carrying a load of 1000 kN. Find the stresses in the concrete and steel bars. Take E for steel = 200 GPa and E for concrete = 14 GPa.*

Data Given:

Area of column = 500 × 500 = 2,50,000 mm², no of steel bars (n) = 4, Diameter of steel bars (d) = 25 mm, Load on column (P) = 1000 kN = 1000 × 10³ N, E_s = 210 GPa, E_c = 14 GPa

We know that area of steel bars,

$$A_s = 4 \times \frac{\pi}{4} \times d^2\ mm^2$$
$$= 4 \times \frac{\pi}{4} \times (25)^2 = 1963\ mm^2$$

∴ Area of concrete,

$$A_c = 250000 - 1963 = 248037\ mm^2$$

We also know that stress in steel,

$$\sigma_s = \frac{E_s}{E_c} \times \sigma_c = \frac{210}{14} \times \sigma_c = 15\ \sigma_c$$

Total load (P),

$$1000 \times 10^3 = (\sigma_s . A_s) + (\sigma_c . A_c)$$
$$\Rightarrow 1000 \times 10^3 = (15\ \sigma_c \times 1963) + (\sigma_c \times 248037) = 277482\ \sigma_c$$
$$\Rightarrow \sigma_c = \frac{1000 \times 10^3}{277482} = 3.6\ N/mm^2 = \mathbf{3.6\ MPa}$$
$$and\ \ \sigma_s = 15\ \sigma_c = 15 \times 3.6 = \mathbf{54\ MPa}$$

1.3 Temperature stress, determine the temperature stress in composite bar (single core):

Temperature Stress:

Whenever there is some increase or decrease in the temperature of a body, it causes the body to expand or contract. If the body is allowed to expand or contract freely, with the rise or fall of the temperature, no stresses are induced in the body. But if the deformation of the body is prevented, some stresses are induced in the body. Such stresses are called temperature stresses. The corresponding strains are called temperature strains.

- ***Define temperature stress or thermal stress.***

  Ans: It is defined as the stress produced due to prevention of elongation or contraction of a bar in order to increase or decrease of temperature.

Temperature Stresses in Simple Bars:

Consider a bar of uniform cross-section is subjected to an increase in temperature.

Let, l = Original length of the bar

t = Increase of temperature

α = Coefficient of linear expansion

The increase in length of the bar due to increase of temperature will be

$$\delta l = l.\alpha.t$$

If this elongation in the bar is prevented by some external force or by fixing the bar ends, the temperature strain (compressive) thus produced will be given by,

$$Temperature\ strain\ (e) = \frac{\delta l}{l} = \frac{l.\alpha.t}{l} = \alpha.t$$

$$\therefore Temperature\ stress\ developed\ (\sigma) = e.E = \alpha.t.E\ (compressive)$$

If the temperature of the bar is lowered, the temperature strain and stress will be tensile in nature.

Note:

- The value of coefficient of linear expansion of materials in everyday use are given below

S. No.	*Material*	*Coefficient of linear expansion/°C (α)*		
1.	Steel	11.5×10^{-6}	to	13×10^{-6}
2.	Wrought Iron, Cast iron	11×10^{-6}	to	12×10^{-6}
3.	Aluminium	23×10^{-6}	to	24×10^{-6}
4.	Copper, Brass, Bronze	17×10^{-6}	to	18×10^{-6}

Temperature Stresses In Composite Bars:

Consider temperature rise of a composite bar consisting of two different materials, one of steel and other of brass, rigidly fastened to each other.

If allowed to expand freely,

Expansion of brass bar $(AB) = l\,\alpha_b\,t$

Expansion of steel bar $(AC) = l\,\alpha_s\,t$

Since coefficient of thermal expansion of brass is greater than that of steel, expansion of brass will be more. But the bars are fastened together and accordingly both will expand to the same final position represented by DD with net expansion of composite system AD equal to δl. To attain this position, brass bar is pushed back and the steel bar is pulled. Obviously compressive stress will be induced in brass bar and tensile stress will be developed in steel bar.

Under equilibrium state,

Compressive force in brass = Tensile force in steel

$\Rightarrow \sigma_b A_b = \sigma_s A_s$

Corresponding to brass bar:

Reduction in elongation,

$$DB = AB - AD = l\,\alpha_b\,t - \delta l$$

$$Strain,\, e_b = \frac{l\,\alpha_b\,t - \delta l}{l} = \alpha_b t - e$$

Where $e = \delta l/l$ is the actual strain of the composite bar.

Corresponding to steel bar:

Extra elongation,

$$CD = AD - AC = \delta l - l\,\alpha_s\,t$$

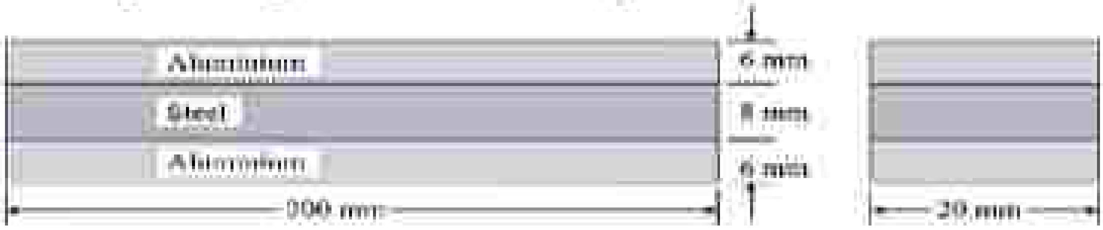
$$Strain,\, e_s = \frac{\delta l - l\,\alpha_s\,t}{l} = e - \alpha_s t$$

Adding e_b and e_s, we get

$$e_b + e_s = (\alpha_b - \alpha_s)t$$

** It may be noted that the nature of the stresses in the bars will get reversed if there is reduction in the temperature of the composite bar.

<u>Example 8</u>: A flat steel bar 200 mm × 20 mm × 8 mm is placed between two aluminium bars 200 mm × 20 mm × 6 mm so as to form a composite bar as shown in Fig.



All the three bars are fastened together at room temperature. Find the stresses in each bar. Where the temperature of the whole assembly is raised through 50°C. Assume:
Young's modulus for steel = 200 GPa
Young's modulus for aluminium = 80 GPa
Coefficient of expansion for steel = 12 × 10⁻⁶/°C
Coefficient of expansion for aluminium = 24 × 10⁻⁶/°C

Solution. Given: Size of steel bar = 200 mm × 20 mm × 8 mm; Size of each aluminium bar = 200 mm × 20 mm × 6 mm; Rise in temperature (t) = 50°C; Young's modulus for steel (E_S) = 200 GPa = 200×10^3 N/mm²; Young's modulus for aluminium (E_A) = 80 GPa = 80×10^3 N/mm²; Coefficient of expansion for steel (α_S) = 12×10^{-6}/°C and coefficient of expansion for aluminium (α_A) = 24×10^{-6}/°C.

Let σ_S = Stress in steel bar and
σ_A = Stress in each aluminium bar.

We know that area of steel bar,

$$A_S = 20 \times 8 = 160\ mm^2$$

and total area of two aluminium bars,

$$A_A = 2 \times 20 \times 6 = 240\ mm^2$$

We also know that when the temperature of the assembly will increase, the free expansion of aluminium bars will be more than that of steel bar (because α_A is more than α_S). Thus, the aluminium bars will be subjected to compressive stress and the steel bar will be subjected to tensile stress. Since the tensile load on the steel bar is equal to the compressive load on the aluminium bars, therefore stress in steel bar,

$$\sigma_S = \frac{A_A}{A_S} \times \sigma_A = \frac{240}{160} \times \sigma_A = 1.5\,\sigma_A$$

We know that strain in steel bar,

$$\varepsilon_S = \frac{\sigma_S}{E_S} = \frac{\sigma_S}{200 \times 10^3}$$

and

$$\varepsilon_A = \frac{\sigma_A}{E_A} = \frac{\sigma_A}{80 \times 10^3}$$

We also know that total strain,

$$\varepsilon_S + \varepsilon_A = t(\alpha_A - \alpha_S)$$

$$\Rightarrow \frac{\sigma_S}{200 \times 10^3} + \frac{\sigma_A}{80 \times 10^3} = 50[(24 \times 10^{-6}) - (12 \times 10^{-6})]$$

$$\Rightarrow \frac{1.5\sigma_A}{200 \times 10^3} + \frac{\sigma_A}{80 \times 10^3} = 50 \times (12 \times 10^{-6})$$

$$\Rightarrow 20 \times 10^{-6}\,\sigma_A = 600 \times 10^{-6}$$

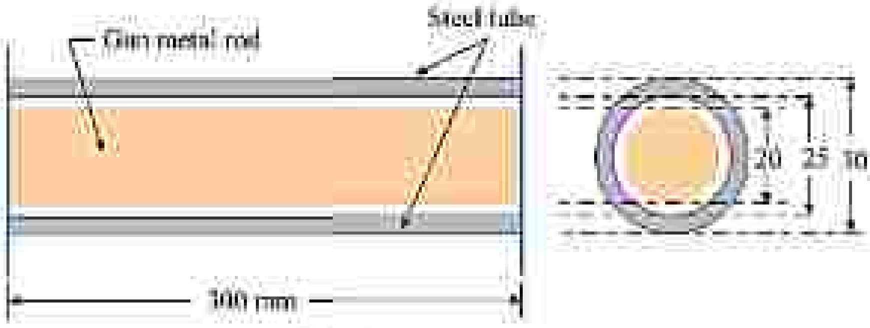
$$\Rightarrow 20\,\sigma_A = 600$$

$$\Rightarrow \sigma_A = \frac{600}{20} = 30\,N/mm^2 = \boldsymbol{30MPa}$$

$\therefore \sigma_S = 1.5\,\sigma_A = 1.5 \times 30 = 45\,N/mm^2 = \mathbf{45 MPa}$

<u>Example 9</u>: A gun metal rod 20 mm diameter, screwed at the ends, passes through a steel tube 25 mm and 30 mm internal and external diameters respectively. The nuts on the rod are screwed tightly home on the ends of the tube. Find the intensity of stress in each metal, when the common temperature rises by 200°F.
Take: Coefficient of expansion for steel = 6 × 10⁻⁶/°F
Coefficient of expansion for gun metal = 10 × 10⁻⁶/°F
Modulus of elasticity for steel = 200 GPa
Modulus of elasticity for gun metal = 100 GPa.

Solution. Given: Diameter of gun metal rod = 20 mm; Internal diameter of steel tube = 25 mm; External diameter of steel tube = 30 mm; Rise in temperature (t) = 200°F; Coefficient of expansion for steel (α_S) = 6×10^{-6} /°F; Coefficient of expansion for gun metals (α_G) = 10×10^{-6}/°F; Modulus of elasticity for steel (E_S) = 200 GPa = 200×10^3 N/mm² and modulus of elasticity for gun metal (E_G) = 100 GPa = 100×10^3 N/mm².



Let σ_G = Stress in gun metal rod, and

σ_S = Stress in steel tube,

We know that area of gun metal rod,

$$A_G = \frac{\pi}{4} \times 20^2 = 100\,\pi\,mm^2$$

And area of steel tube

$$A_S = \frac{\pi}{4} \times [(30)^2 - (25)^2] = 68.75\,\pi\,mm^2$$

We also know that when the common temperature of the gun metal rod and steel tube will increase, the free expansion of gun metal rod will be more than that of steel tube (because α_G is greater than α_S). Thus, the gun metal rod will be subjected to compressive stress and the steel tube will be subjected to tensile stress. Since the tensile load on the steel tube is equal to the compressive load on the gun metal rod, therefore stress in steel,

$$\sigma_S = \frac{A_G}{A_S} \times \sigma_G = \frac{100\,\pi}{68.75\,\pi} \times \sigma_G = 1.45\,\sigma_G$$

We know that strain in steel tube,

$$e_S = \frac{\sigma_S}{E_S} = \frac{\sigma_S}{200 \times 10^3}$$

$$\varepsilon_c = \frac{\sigma_c}{E_c} = \frac{\sigma_c}{100 \times 10^3}$$

We also know that total strain,

$$\varepsilon_s + \varepsilon_c = t(\alpha_c - \alpha_s)$$

$$\Rightarrow \frac{\sigma_s}{200 \times 10^3} + \frac{\sigma_c}{100 \times 10^3} = 200[(10 \times 10^{-6}) - (6 \times 10^{-6})]$$

$$\Rightarrow \frac{1.45\sigma_c}{200 \times 10^3} + \frac{\sigma_c}{100 \times 10^3} = 200 \times (4 \times 10^{-6})$$

$$\Rightarrow \frac{3.45\,\sigma_c}{200 \times 10^3} = 800 \times 10^{-6}$$

$$\Rightarrow 3.45\sigma_c = (800 \times 10^{-6}) \times (200 \times 10^3)$$

$$\Rightarrow \sigma_c = \frac{160}{3.45} = 46.4\,N/mm^2 = \mathbf{46.4\,MPa}$$

$$\therefore \sigma_s = 1.45\,\sigma_c = 1.45 \times 46.4 = 67.3\,N/mm^2 = \mathbf{67.3\,MPa}$$

1.4 <u>Strain energy and resilience, Stress due to gradually applied, suddenly applied and impact load:</u>

<u>Strain Energy</u>:

- When an elastic body is loaded within elastic limit, it deforms and some work is done which is stored within the body in the form of internal energy. This stored energy in the deformed body is known as strain energy or potential energy of deformation and is denoted by '*U*'.

<u>Resilience</u>:

- It is the ability of a material to regain its original shape on removal of the applied load. It is defined as the strain energy per unit volume in simple tension or compression and is equal to $\sigma^2/2E$. It is also referred as strain energy density and is denoted by '**u**'.

<u>Stresses due to different types of loads</u>:

A body may be subjected to following types of loads:

1. Gradually applied loads
2. Suddenly applied loads
3. Falling or impact loads

1. <u>Gradually applied loads</u>:

A body is said to be acted upon by a gradually applied load if the load increases from zero and reaches its final value stepwise.

Now consider a metallic bar subjected to a gradual load.

Let, P = Load gradually applied

A = Cross sectional area of the bar

l = Length of the bar

E = Modulus of elasticity of the bar material

δl = Deformation of the bar due to load

Since the load applied is gradual, and varies from zero to P, therefore the average load is equal to P/2

$$\therefore Work\ done = Force \times Distance$$

$$= Average\ load \times Deformation$$

$$= \frac{P}{2} \times \delta l = \frac{P}{2}(e.l)$$

$$= \frac{1}{2}\sigma.e.A.l$$

$$= \frac{1}{2} \times (stress \times strain \times volume)$$

$$= \frac{1}{2} \times \sigma.\frac{\sigma}{E}.A\,l$$

$$= \frac{1}{2} \times \frac{\sigma^2}{E} \times A\,l$$

$$= \frac{\sigma^2}{2E} \times V$$

Since the energy stored is also equal to the work done, therefore strain energy stored,

$$U = \frac{\sigma^2}{2E} \times V$$

We also know that resilience = strain energy per unit volume

$$= \sigma^2/2E$$

<u>Example 10</u>: Calculate the strain energy stored in a bar 2 m long, 50 mm wide and 40 mm thick when it is subjected to a tensile load of 60kN. Take E as 200 GPa.

Solution. Given: Length of bar (l) = 2 m = 2×10^3 mm; Width of bar (b) = 50 mm; Thickness of bar (t) = 40 mm; Tensile load on bar (P) = 60 kN = 60×10^3 N and modulus of elasticity (E) = 200 GPa = 200×10^3 N/mm^2.

We know that stress in the bar,

$$\sigma = \frac{P}{A} = \frac{60 \times 10^3}{50 \times 40} = 30\ N/mm^2$$

$\therefore$Strain energy stored in the bar,

$$U = \frac{\sigma^2}{2E} \times V = \frac{30^2}{2 \times (200 \times 10^3)} \times 4 \times 10^6\ N-mm$$

$$\Rightarrow U = 9 \times 10^3\ N-mm = 9\ kN-mm$$

2. <u>Suddenly applied loads</u>:

When the load is applied all of a sudden and not stepwise is called suddenly applied load.

Now consider a bar subjected to a sudden load.

Let, P = Load applied suddenly

A = Cross sectional area of the bar

l = Length of the bar

E = Modulus of elasticity of the bar material

δl = Deformation of the bar due to load

σ = Stress induced by the application of the sudden load

Since the load is applied suddenly, therefore the load (P) is constant throughout the process of deformation of the bar.

$$\therefore Work\ done = Force \times Distance$$
$$= load \times Deformation$$
$$= P \times \delta l$$

We know that strain energy stored,

$$U = \frac{\sigma^2}{2E} \times A\,l$$

Since the energy stored is equal to work done, therefore

$$\frac{\sigma^2}{2E} \times A\,l = P \times \delta l = P \times \frac{\sigma}{E}\,l$$
$$\Rightarrow \sigma = 2 \times \frac{P}{A}$$

***Example 11**: An axial pull of 20 kN is suddenly applied on a steel rod 2.5 m long and 1000 mm² in cross-section. Calculate the strain energy, which can be absorbed in the rod. Take E = 200 GPa.*

Solution. Given: Axial pull on the rod (P) = 20 kN = 20×10^3 N; Length of rod (l) = 2.5 m = 2.5×10^3 mm; Cross-sectional area of rod (A) =1000 mm² and modulus of elasticity (E) = 200 GPa = 200×10^3 N/mm².

We know that stress in the rod, when the load is suddenly applied,

$$\sigma = 2 \times \frac{P}{A} = 2 \times \frac{20 \times 10^3}{1000} = 40\ N/mm^2$$

and volume of the rod,

$$V = l \cdot A = (2.5 \times 10^3) \times 1000 = 2.5 \times 10^6\ mm^3$$

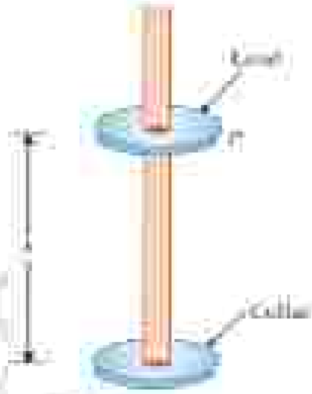
Strain energy which can be absorbed in the rod,

$$U = \frac{\sigma^2}{2E} \times V = \frac{40^2}{2 \times (200 \times 10^3)} \times (2.5 \times 10^6)\ N-mm$$

$$\Rightarrow U = 10 \times 10^3\ N-mm = \mathbf{10\ kN-mm}$$

3. Falling or impact loads:

- The load which falls from a height or strike the body with certain momentum is called falling or impact load.
- Now consider a bar subject to a load applied with impact as shown in Figure.



Let, P = Load applied with impact

A = Cross sectional area of the bar

l = Length of the bar

E = Modulus of elasticity of the bar material

δl = Deformation of the bar due to load

σ = Stress induced by the application of this load with impact

h = Height through which the load will fall, before impacting on the collar

$$\therefore Work\ done = Load \times Distance\ moved$$
$$= P \times (h + \delta l)$$

- And energy stored,

$$U = \frac{\sigma^2}{2E} \times A\,l$$

- Since energy stored is equal to the work done, therefore

$$\frac{\sigma^2}{2E} \times A\,l = P(h + \delta l) = P\left(h + \frac{\sigma}{E}.l\right)$$
$$\Rightarrow \frac{\sigma^2}{2E} \times A\,l = Ph + \frac{P\sigma l}{E}$$
$$\therefore \sigma^2\left(\frac{Al}{2E}\right) - \sigma\left(\frac{Pl}{E}\right) - Ph = 0$$

- Multiplying both sides by (E/Al)

$$\frac{\sigma^2}{2} - \sigma\left(\frac{P}{A}\right) - \frac{PEh}{Al} = 0$$

- This is a quadratic equation, we know that

$$\sigma = \frac{P}{A} \pm \sqrt{\left(\frac{P}{A}\right)^2 + \left(4 \times \frac{1}{2}\right)\left(\frac{PEh}{Al}\right)}$$
$$= \frac{P}{A}\left[1 \pm \sqrt{1 + \frac{2AEh}{Pl}}\right]$$

- Once the stress (σ) is obtained, the corresponding instantaneous deformation (δl) or the strain energy stored may be found out as usual.
- <u>Note</u>: When δl is very small as compared to h, then

$$Work\ done = Ph$$

$$\Rightarrow \frac{\sigma^2}{2E} Al = Ph$$

$$\Rightarrow \sigma^2 = \frac{2EPh}{Al}$$

$$\therefore \sigma = \sqrt{\frac{2EPh}{Al}}$$

Example 12: ***A 2 m long alloy bar of 1500 mm² cross-sectional area hangs vertically and has a collar securely fixed at its lower end. Find the stress induced in the bar, when a weight of 2 kN falls from a height of 100 mm on the collar. Take E = 120 GPa. Also find the strain energy stored in the bar.***

Solution. Given: Length of bar (l) = 2 m = 2×10^3 mm, Cross-sectional area of bar (A) = 1500 mm², Weight falling on collar of bar (P) = 2 kN = 2×10^3 N; Height from which weight falls (h) = 100 mm and modulus of elasticity (E) = 120 GPa = 120×10^3 N/mm².

Stress induced in the bar

We know that in this case, extension of the bar will be small and negligible as compared to the height (h) from where the weight falls on the collar (due to small value of weight i.e., 2 kN and a large value of h i.e., 100 mm). Therefore, stress induced in the bar

$$\sigma = \sqrt{\frac{2\,E\,P\,h}{A\,.\,l}} = \sqrt{\frac{2 \times (120 \times 10^3) \times (2 \times 10^3) \times 100}{1500 \times (2 \times 10^3)}}\ N/mm^2$$

$$\Rightarrow \sigma = 126.5\ N/mm^2 = \mathbf{126.5\ MPa}$$

Strain energy stored in the bar

We also know that volume of the bar,

$$V = l\,.\,A = (2 \times 10^3) \times 1500 = 3 \times 10^6\ mm^3$$

and strain energy stored in the bar,

$$U = \frac{\sigma^2}{2E} \times V = \frac{(126.5)^2}{2 \times (120 \times 10^3)} \times (3 \times 10^6)\ N - mm$$

$$\Rightarrow U = 200 \times 10^3\ N - mm = \mathbf{200\ N - m}$$

POSSIBLE SHORT TYPE QUESTIONS WITH ANSWER:

1. Define stress and state its S.I unit.

- The internal resistance per unit area offered by the material of the body against external loading is called intensity of stress or simply called as stress.

Or

- The internal resistance which the body offers to meet with the load is called stress.
- **Mathematically,**

$$\sigma = \frac{R}{A} = \frac{P}{A}$$

- Its unit is N/m^2, N/mm^2, KPa, MPa, GPa.

2. State Hooke's Law. (W – 2019)

- Hooke's law states that when a material is loaded within elastic limit, stress is directly proportional to strain.
- Mathematically,

$$stress \propto strain$$

$$\Rightarrow \frac{stress}{strain} = constant\ (E)$$

Where the constant of proportionality ***E*** is called ***Young's modulus or modulus of elasticity***.

3. Define Strain.

- The strain is the deformation produced by stress.
- The ratio of change in dimension to original dimension of a body is called as strain.
- It is denoted by the letter 'e' or 'ε'.
- It is a unit less quantity.
- Strain(e) = $\frac{change\ in\ dimension}{original\ dimension}$

4. Define Poisson's ratio.

- If a body is stressed within its elastic limit, the lateral strain bears a constant ratio to the linear strain this contant is known as Poisson's ratio.
- It is denoted by symbol 'μ' or 1/m.
- It is unit less.
- Mathematically,

$$\mu = \frac{lateral\ strain}{linear\ strain} = \frac{1}{m}$$

The value of 'μ' lies from 0.25 to 0.34 for different materials.

5. Define young's modulus of elasticity.

- It is defined as, "The ratio of stress to strain".
- It is denoted by the letter '*E*'.
- Mathematically,

$$E = \frac{stress}{strain} = \frac{\sigma}{e}$$

Its unit is same as stress i.e., N/m^2, N/mm^2, KPa, MPa, GPa.

6. What is meant by modulus of rigidity? (W – 2019)

- It is defined as, "The ratio of shear stress to shear stain".
- It is denoted by letter ***C, N* or *G***.
- Mathematically,

$$C = \frac{shear\ stress}{shear\ strain} = \frac{\tau}{e_s}$$

- Its unit is N/m^2, N/mm^2, KPa, MPa, GPa.

7. Define temperature stress. (W - 2020)

- It is defined as the stress produced due to prevention of elongation or contraction of a bar in order to increase or decrease of temperature.

8. What is the difference between stress and strain? (W – 2019, 2020)

- The main difference between stress and strain is that stress measures the deforming force per unit area of the object, whereas strain measures the relative change in length caused by a deforming force.
- Stress is measured in Pascal (Pa) but strain has no unit, it is simply a ratio.

9. Define strain energy.

- When an elastic body is loaded within elastic limit, it deforms and some work is done which is stored within the body in the form of internal energy. This stored energy in the deformed body is known as strain energy or potential energy of deformation and is denoted by '*U*'.

10. What is resilience? (W – 2019, 2020)

- It is the ability of a material to regain its original shape on removal of the applied load. It is defined as the strain energy per unit volume in simple tension or compression and is equal to $\sigma^2/2E$. It is also referred as strain energy density and is denoted by '*u*'.

11. Write down the expression for strain energy.[W-2021]

- $U = \frac{\sigma^2}{2E} \times V$

  σ =Normal stress induced from the body

  E = *Youngs modulus of the material of the body*

  V=Volume of the body

POSSIBLE LONG TYPE QUESTIONS:

1. A steel bar 2 m long and 150 mm^2 in cross section is subjected to an axial pull of 15 KN. Find the elongation of the bar. Take E = 200GPa. (W – 2019)

Hints: Use the formula, $\delta l = \frac{Pl}{AE}$ and refer example – 1

2. Derive the relationship between Young's Modulus of Elasticity & Bulk Modulus. (W – 2019)

Hints: Refer page no – 10

3. A rectangular body 400 mm long, 100 mm wide & 50 mm thick is subjected to a shear stress of 60 MPa. Determine the strain energy stored in the body. Take modulus of rigidity = 80 N/mm². **(W – 2019)**

Hints: Use the formula, $U = \frac{\tau^2}{2C} \times V$

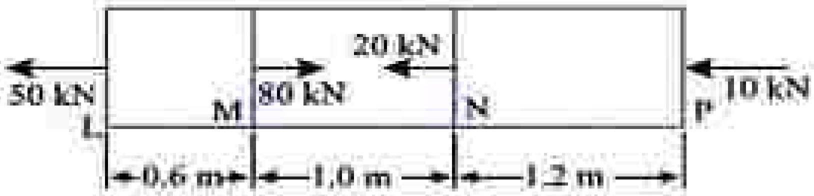
4. A rod 150 cm long and of diameter 2.0 cm is subjected to an axial pull of 20 KN. If the modulus of elasticity of the material of the rod is 2×10^5 N/mm², determine (i) the stress (ii) the strain (iii) the elongation of the rod. **(W – 2020)**

Hints: Use the formula, $= \frac{P}{A}$, $e = \frac{\sigma}{E}$, $\delta l = \frac{Pl}{AE}$ or $\delta l = e \times l$

5. Derive the relationship between Young's Modulus of Elasticity & Modulus of Rigidity. **(W – 2020)**

Hints: Refer page no – 09

6. A brass bar having cross sectional area of 1000 mm² is subjected to axial forces shown in the figure. Find the total elongation of the bar. Modulus of elasticity of brass is 100 GN/m². **(W – 2020)**



Hints: Use the formula,

$$\delta l = \frac{P_1 l_1}{AE} + \frac{P_2 l_2}{AE} + \frac{P_3 l_3}{AE} \ldots\ldots$$

7. A reinforced short concrete column 250 mm × 250 mm in section is reinforced with 8 steel bars. The total area of steel bars is 2500 mm². The column carries a load of 390 KN. If the modulus of elasticity for steel is 15 times that of concrete, find the stresses in concrete and steel. **(W – 2020)**

Hints: Refer page no – 15 (Example – 7)

8. A steel bar ABC of 400 mm length and 20 mm diameter is subjected to a point loads as shown in Figure. Determine the total change in length of the bar. Take E = 200 GPa.

[Ans: 0.32 mm]

9. A reinforced concrete column of 300 mm diameter contains 4 bars of 22 mm diameter. Find the total load, the column can carry, if the stresses in steel and concrete are 50 MPa and 3 MPa respectively.

[Ans. 283.5 KN]

10. An aluminium rod of 20 mm diameter is completely enclosed in a steel tube of 30 mm external diameter and both the ends of the assembly are rigidly connected. If the composite bar is heated through 50°C, find the stresses developed in the aluminium rod and steel tube. Take:

Modulus of elasticity for steel = 200 GPa

Modulus of elasticity for aluminium = 80 GPa

Coefficient of expansion for steel = $12 \times 10^{-6}/°C$

Coefficient of expansion for aluminium = $18 \times 10^{-6}/°C$

[Ans. 14.5 MPa (Comp.); 18.1 MPa (Tension)]

11. A mild steel rod 1 m long and 20 mm diameter is subjected to an axial pull of 62.5 kN. What is the elongation of the rod, when the load is applied (i) gradually and (ii) suddenly? Take E as 200 GPa.

[Ans. 1mm; 2mm]

11. Explain temperature stress and derive its expression.[W-2021]

12.A steel rod 22m in diameter and 1.5m long is subjected to an axial pull of 35KN. Find(1)The intensity of stress.(2)The strain and elongation.Take $E=2*100000N/MM^2$. [W-2021]

13.A hollow cylinder 2m long has an outside diameter of 50mm and inside diameter of 50mm and inside diameter of 30mm.If the cylinder is carrying a load of 25KN, find the stress in the cylinder.Also find the deformation of the cylinder, if the modulus of elasticity for the cylinder material is 100GPA.[W-2022]

14.Prove- E=3K(1-2/m) or E=3K[1-2/M], where E=Young's modulus, 1/m=Poisson ratio.[W-2022]

CHAPTER NO. – 02

THIN CYLINDER AND SPHERICAL SHELL UNDER INTERNAL PRESSURE

LEARNING OBJECTIVES:

2.1 Definition of hoop and longitudinal stress, strain
2.2 Derivation of hoop stress, longitudinal stress, hoop strain, Longitudinal strain and volumetric strain
2.3 Computation of the change in length, diameter and volume
2.4 Simple problems on above

INTRODUCTION:

- Thin pressure vessels or shells are used to carry fluid. The thickness of the wall is small as compared to the diameter. These are made by rolling the sheet metal and joining the ends by riveting or welding. Example: boiler shell, gas cylinder, water tank, pipe line carrying fluid under pressure.
- A pressure vessel is a thin one if the ratio of its internal diameter to wall thickness ≥ 20, i.e.;

$$\frac{d}{t} \geq 20 \quad or \quad \frac{t}{d} \leq \frac{1}{20}$$

Otherwise it is a thick shell.
- Shape of shells may be cylindrical, spherical, cylindrical shell with hemispherical ends.

2.1 Definition Of Hoop And Longitudinal Stress, Strain

Stresses in a Thin Cylindrical Shell:

- Whenever a cylindrical shell is subjected to an internal pressure, its walls are subjected to tensile stresses. The walls of the cylindrical shell will be subjected to the following types of stresses.
  1. Circumferential stress and
  2. Longitudinal stress
  3. Radial stress

Hoop stress or circumferential stress:

- It is a tensile stress acting along the circumference of the cylinder.
- It is denotes by (σ_c).

Hoop strain:

- Circumference depends on the diameter of the shell. So it is the ratio of change in diameter to the original diameter of the shell.
- Mathematically,

$$e_c = \frac{\delta d}{d}$$

Longitudinal stress:

- It is a tensile stress acting along the length of the cylinder. It develops only if cylinder has closed ends.
- It is denoted by (σ_l).

Longitudinal strain:

- It is the ratio of change in length to the original length of the shell.
- Mathematically,

$$e_l = \frac{\delta l}{l}$$

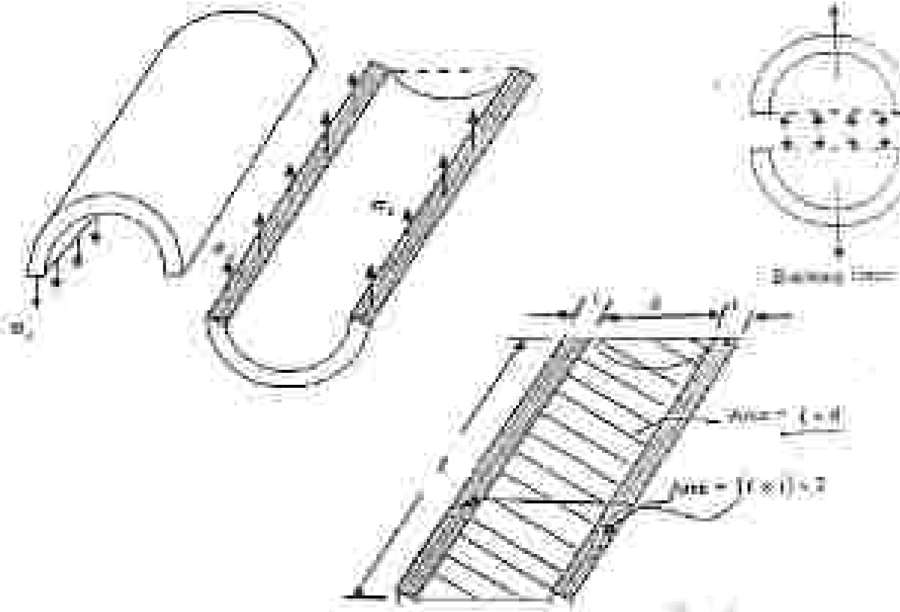
Radial stress:

- It is a compressive stress acting along the radius of the cylinder. It is small and neglected.
- It is denoted by (σ_r).

2.2 Derivation of hoop stress, longitudinal stress, hoop strain, longitudinal strain and volumetric strain

Derivation of Hoop Stress in a Cylindrical Shell:

Consider a thin cylindrical shell subjected to an internal pressure.



Let P = Internal fluid pressure inside the cylinder
σ_c = Circumferential or Hoop stress
d = Internal diameter of the cylinder
t = Thickness of the cylinder
l = Length of the cylinder

Bursting force = internal fluid pressure × area
= $P \times l \times d$
Resisting force = circumferential stress × area on which it acts
= $\sigma_c \times (2 \times l \times t)$

Considering the equilibrium of a half of the cylinder

Resisting force = bursting force

$$\Rightarrow \sigma_c \times (2 \times l \times t) = P \times l \times d$$

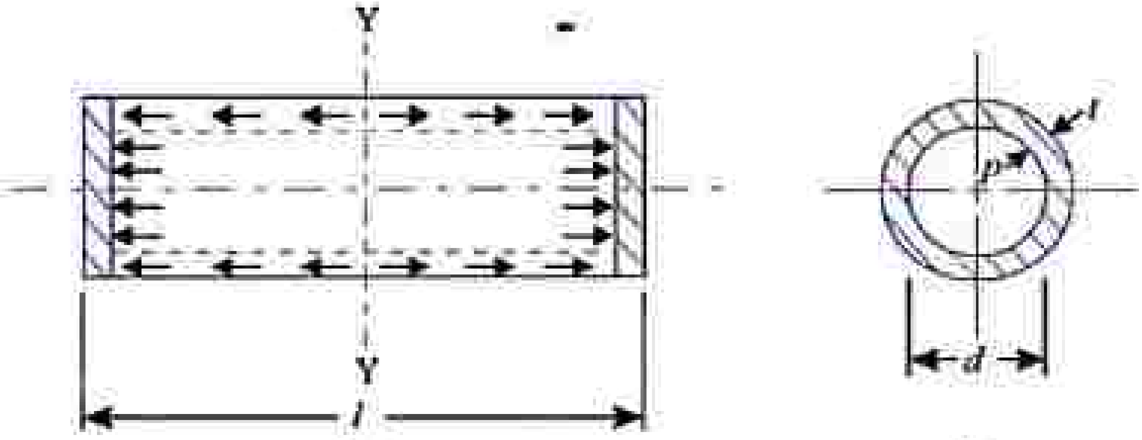
$$\Rightarrow \sigma_c = \frac{Pd}{2t}$$

Note: if η_l is the efficiency of longitudinal joints of the shell, then hoop stress

$$\sigma_c = \frac{Pd}{2t\eta_l}$$

Derivation of Longitudinal Stress in a Cylindrical Shell:

Let P = Internal fluid pressure
σ_l = Circumferential or Hoop stress
d = Internal diameter of the cylinder
t = Thickness of the cylinder
l = Length of the cylinder



(a) (b)

Bursting force = internal fluid pressure × area

$$= P \times \frac{\pi}{4} \times d^2$$

Resisting force = longitudinal stress × area on which it acts

$$= \sigma_l \times \pi \times dt$$

For equilibrium,

Resisting force = bursting force

$$\Rightarrow \sigma_l \times \pi \times dt = P \times \frac{\pi}{4} \times d^2$$

$$\Rightarrow \sigma_l = \frac{Pd}{4t}$$

Note 1: If η_c is the efficiency of circumferential joints of the shell, then longitudinal stress

$$\sigma_l = \frac{Pd}{2t\eta_c}$$

Note 2: In case of thin cylinder subjected to internal fluid pressure, the hoop stress developed is twice that of the longitudinal stress.

$$\sigma_c = 2\sigma_l$$

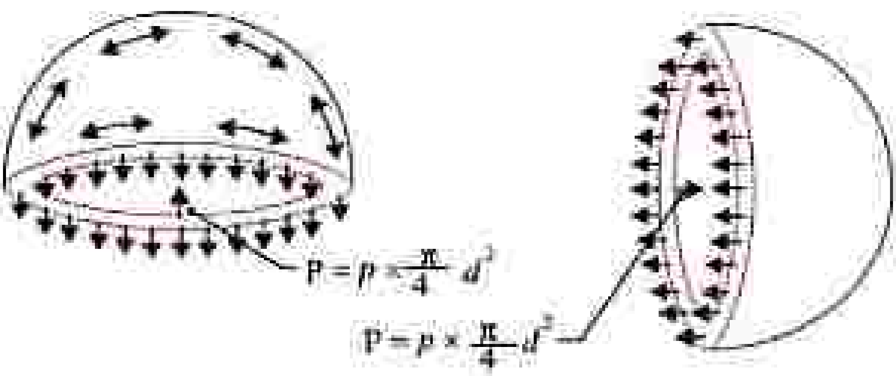
Derivation of Hoop Stress in a Spherical Shell:

Let P = Internal fluid pressure inside the shell

σ_c = Circumferential or Hoop stress

d = Internal diameter of the shell

t = Thickness of the shell



Bursting force = internal fluid pressure × area

$$= P \times \frac{\pi}{4} \times d^2$$

Resisting force = circumferential stress × area on which it acts

$$= \sigma_c \times \pi \times dt$$

For equilibrium

Resisting force = bursting force

$$\Rightarrow \sigma_c \times \pi \times dt = P \times \frac{\pi}{4} \times d^2$$

$$\Rightarrow \sigma_c = \frac{Pd}{4t}$$

Note 1: If η_c is the efficiency of the circumferential joints of the spherical shell, then hoop stress,

$$\sigma_c = \frac{Pd}{4t\eta_c}$$

Note 2: There is no longitudinal stress in spherical shells so σ_c (Hoop stress) is the only stress induced in the spherical shells.

Hoop Strain, Longitudinal Strain and Volumetric Strain:

Let l = Length of the shell

d = Diameter of the shell

t = Thickness of the shell

P = Intensity of pressure

μ = 1/m = poisson's ratio

σ_c = Hoop stress or circumferential stress

σ_l = Longitudinal stress

Direct strain (e_c) due to $\sigma_c = \frac{\sigma_c}{E}$

Direct strain (e_l) due to $\sigma_l = \frac{\sigma_l}{E}$

For cylindrical shells:

- **Net Circumferential strain or hoop strain,**

$$e_c = \frac{\sigma_c}{E} - \frac{\sigma_l}{mE}$$

$$= \frac{Pd}{2tE} - \frac{1}{m} \times \frac{Pd}{4tE}$$

$$= \frac{Pd}{2tE}\left(1 - \frac{1}{2m}\right)$$

- **Net Longitudinal strain,**

$$e_l = \frac{\sigma_l}{E} - \frac{\sigma_c}{mE}$$

$$= \frac{Pd}{4tE} - \frac{1}{m} \times \frac{Pd}{2tE}$$

$$= \frac{Pd}{4tE}\left(1 - \frac{2}{m}\right)$$

- **Volumetric strain,**

$$e_v = \text{Algebric sum of net strains in all axes}$$

$$= \text{net longitudinal strain} + 2 \times \text{net hoop strain}$$

$$= e_l + 2e_c$$

For spherical shells:

- **Circumferential strain or hoop strain,**

$$e_c = \frac{\sigma_c}{E} - \frac{\sigma_c}{mE}$$
$$= \frac{\sigma_c}{E}\left(1 - \frac{1}{m}\right)$$
$$= \frac{Pd}{4tE}\left(1 - \frac{1}{m}\right)$$

- **Volumetric strain,**

$$e_v = Algebric\ sum\ of\ strains\ in\ all\ the\ three\ axis$$
$$= e_c + e_c + e_c = 3e_c$$

2.3 Computation of the change in length, diameter and volume

For cylindrical shells:

- **Change in length:**

  Change in length depends upon longitudinal strain.

$$e_l = \frac{\delta l}{l}$$
$$\Rightarrow \delta l = e_l \times l$$
$$= \frac{Pd}{4tE}\left(1 - \frac{2}{m}\right) \times l$$
$$= \frac{Pdl}{4tE}\left(1 - \frac{2}{m}\right)$$

- **Change in diameter:**

  Change in diameter depends upon circumferential or hoop strain.

$$e_c = \frac{\delta d}{d}$$
$$\Rightarrow \delta d = e_c \times d$$
$$= \frac{Pd}{2tE}\left(1 - \frac{1}{2m}\right) \times d$$
$$= \frac{Pd^2}{2tE}\left(1 - \frac{1}{2m}\right)$$

- **Change in volume:**

  Change in volume depends upon volumetric strain.

$$e_v = \frac{\delta V}{V}$$
$$\Rightarrow \delta V = e_v \times V$$
$$= (e_l + 2e_c) \times V$$
$$= \left[\frac{Pd}{4tE}\left(1 - \frac{2}{m}\right) + 2 \times \frac{Pd}{2tE}\left(1 - \frac{1}{2m}\right)\right] \times V$$
$$= \frac{Pd}{2tE}\left[\left(\frac{1}{2} - \frac{1}{m}\right) + \left(2 - \frac{1}{m}\right)\right] \times V$$
$$= \frac{PdV}{2tE}\left(\frac{5}{2} - \frac{2}{m}\right)$$

$\left[Where, V = \frac{\pi}{4} d^2 l\right]$

For spherical shells:

- **Change in diameter:**
  Change in diameter depends upon circumferential or hoop strain.

$$e_c = \frac{\delta d}{d}$$
$$\Rightarrow \delta d = e_c \times d$$
$$= \frac{Pd}{4tE}\left(1 - \frac{1}{m}\right) \times d$$
$$= \frac{Pd^2}{4tE}\left(1 - \frac{1}{m}\right)$$

- **Change in volume:**
  Change in volume depends upon volumetric strain.

$$e_v = \frac{\delta V}{V}$$
$$\Rightarrow \delta V = e_v \times V$$
$$= 3e_c \times V$$
$$= \frac{3Pd}{4tE}\left(1 - \frac{1}{m}\right) \times V$$

$$\left[Where, V = \frac{4}{3}\pi r^3 \quad or \quad \frac{\pi d^3}{6}\right]$$

2.4 Simple problems on above:

Example 1: A stream boiler of 800 mm diameter is made up of 10 mm thick plates. If the boiler is subjected to an internal pressure of 2.5 MPa, find the circumferential and longitudinal stresses induced in the boiler plates.

Solution. Given: Diameter of boiler (d) = 800 mm; Thickness of plates (t) = 10 mm and internal pressure (p) = 2.5 MPa = 2.5 N/mm².

Circumferential stress induced in the boiler plates

We know that circumferential stress induced in the boiler plates,

$$\sigma_c = \frac{pd}{2t} = \frac{2.5 \times 800}{2 \times 10} = 100\, N/mm^2 = \mathbf{100\, MPa}$$

Longitudinal stress induced in the boiler plates

We also know that longitudinal stress induced in the boiler plates,

$$\sigma_l = \frac{pd}{4t} = \frac{2.5 \times 800}{4 \times 10} = 50\, N/mm^2 = \mathbf{50\, MPa}$$

Example 2: A cylindrical shell of 1.3 m diameter is made up of 18 mm thick plates. Find the circumferential and longitudinal stress in the plates, if the boiler is subjected to an internal pressure of 2.4 MPa. Take efficiency of the joints as 70%.

Solution. Given: Diameter of shell (d) = 1.3 m = 1.3×10^3 mm; Thickness of plates (t) = 18 mm; Internal pressure (p) = 2.4 MPa = 2.4 N/mm² and efficiency (η) = 70% = 0.7.

Circumferential stress

We know that circumferential stress,

$$\sigma_c = \frac{pd}{2t\eta} = \frac{2.4 \times (1.3 \times 10^3)}{2 \times 18 \times 0.7} = 124\, N/mm^2 = \mathbf{124\, MPa}$$

Longitudinal stress

We also know that longitudinal stress,

$$\sigma_l = \frac{pd}{4t\eta} = \frac{2.4 \times (1.3 \times 10^3)}{4 \times 18 \times 0.7} = 62\,N/mm^2 = \mathbf{62\,MPa}$$

Example 3: A cylindrical thin drum 800 mm in diameter and 4 m long is made of 10 mm thick plates. If the drum is subjected to an internal pressure of 2.5 MPa, determine its changes in diameter and length. Take E as 200 GPa and Poisson's ratio as 0.25.

Solution. Given: Diameter of drum (d) = 800 mm; Length of drum (l) = 4 m = 4 × 10^3 mm; Thickness of plates (t) = 10 mm; Internal pressure (p) = 2.5 MPa = 2.5 N/mm^2; Modulus of elasticity (E) = 200 GPa = 200 × 10^3 N/mm^2 and poisson's ratio (1/m) = 0.25.

Change in diameter

We know that change in diameter,

$$\delta d = \frac{pd^2}{2tE}\left(1 - \frac{1}{2m}\right) = \frac{2.5 \times 800^2}{2 \times 10 \times (200 \times 10^3)}\left(1 - \frac{0.25}{2}\right) = \mathbf{0.35\,mm}$$

Change in length

We also know that change in length,

$$\delta l = \frac{pdl}{2tE}\left(\frac{1}{2} - \frac{1}{m}\right) = \frac{2.5 \times 800 \times (4 \times 10^3)}{2 \times 10 \times (200 \times 10^3)}\left(\frac{1}{2} - 0.25\right) = \mathbf{0.5\,mm}$$

Example 4: A spherical gas vessel of 1.2 m diameter is subjected to a pressure of 1.8 MPa. Determine the stress induced in the vessel plate, if its thickness is 5 mm.

Solution. Given: Diameter of vessel (d) = 1.2 m = 1.2 × 10^3 mm; Internal pressure (p) = 1.8 MPa = 1.8 N/mm^2 and thickness of plates (t) = 5 mm.

We know that stress in the vessel plates,

$$\sigma = \frac{pd}{4t} = \frac{1.8 \times (1.2 \times 10^3)}{4 \times 5} = 108\,N/mm^2 = \mathbf{108\,MPa}$$

POSSIBLE SHORT TYPE QUESTIONS WITH ANSWER:

1. Define hoop stress. (W – 2019, 2020)

- It is a tensile stress acting along the circumference of the cylinder.
- It is denoted by (σ_c).

2. Define longitudinal stress. (W – 2020)

- It is a tensile stress acting along the length of the cylinder. It develops only if cylinder has closed ends.
- It is denoted by (σ_l).

3. Define principal stress and its uses.[W-2020],[W-2022]

- At any point in a strained material, there are 3-planes,mutually perpendicular to each other, which carry direct stress only and no shear stress. These plain are known as principal plane. The magnitude of direct stresses across a principal plane is called principal stress.

4. Write the significance of mohr's circle.[W-2021]

- The construction of mohr's circle of stresses as well as determination of normal, shear and resultant stresses is very easier than the analytical method.

5. State the application of a thin cylinder shell.(W-2022)

Ans: These cylinder are generally used in pipes, pressure shells and boilers.

POSSIBLE LONG TYPE QUESTIONS:

1. A cylindrical shell 2 m long and 1 m internal diameter is made up of 20 mm thick plates. Find the circumferential stress and longitudinal stress in the shell material, if it is subjected to an internal pressure of 5 MPa. **(W – 2019)**

Hints: Refer example 01

2. Derive the expression for hoop stress and longitudinal stress in case of a thin cylindrical shell. **(W – 2020) (W-2021)**

Hints: Refer article no 2.2

3. A steam boiler of 1.25 m in diameter is subjected to an internal pressure of 1.6 MPa. If the steam boiler is made up of 20 mm thick plates, calculate the circumferential and longitudinal stresses. Take efficiency of the circumferential and longitudinal joints as 75% and 60% respectively.

Hints: Refer example 02

4. The principal stress at a point in a bar are 150N/MM^2(tensile) and 80N/MM^2(compressive) Determine the resultant stress in magnitude and direction on a plane inclind at 60^0 to the axis of the major principal stress. Also, find the maximum intensity of shear stress in the material at that point.(w-21)

5.The stresses at point of a machine component are 150MPa and 50MPa both tensile. Find the intensities of normal, shear and resultant stresses on a plane inclined at an angle of 55^0 with the axis of major tensile stress.(w-22)

6.Derive a formula for the longitudinal stress in a thin cylindrical shell subjected to an internal pressure .(W-2022)

7.Acylindrical shell of 1.3M dimeter is made up of 18MM thick plates. Find the circumferential and longitudinal stress in the stress, if the boiler is subjected to and internal pressure of 2.4MPa. Take efficiency of the joint as 70%.(W-2022)

CHAPTER NO. – 03

TWO - DIMENSIONAL STRESS SYSTEMS

LEARNING OBJECTIVES:

3.1 Determination of normal stress, shear stress and resultant stress on oblique plane
3.2 Location of principal plane and computation of principal stress
3.3 Location of principal plane and computation of principal stress and Maximum shear stress using Mohr's circle

Introduction:

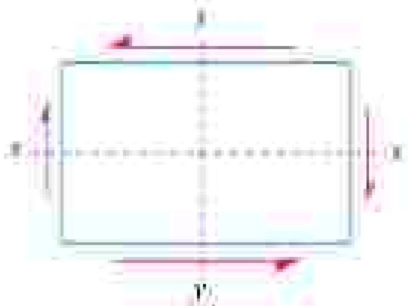
- Attention was focused in previous chapters to study the effect of simple stresses which were either normal or tangential acting on a particular plane. The subject matter included in this chapter deals with the analytical and graphical methods to investigate the state and intensity of stresses on an arbitrary oblique plane through an element in a body being acted upon by complex loading conditions.

Methods for the Stresses on an Oblique Section of a Body:

- The following two methods for the determination of stresses on an oblique section of a strained body are important from the subject point of view:
  1. Analytical method and 2. Graphical method.

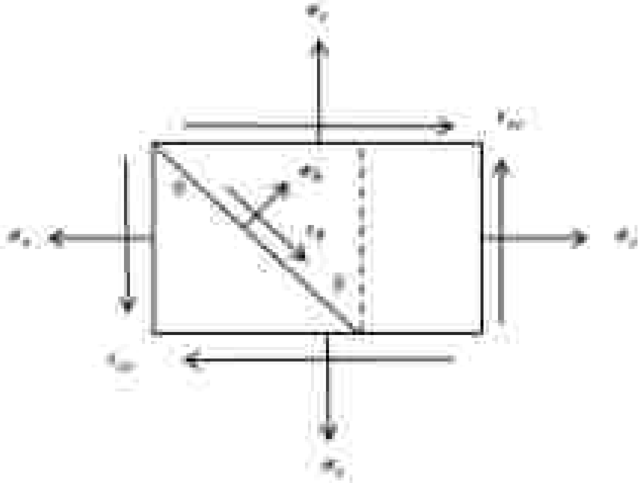
Sign Conventions for Analytical Method:

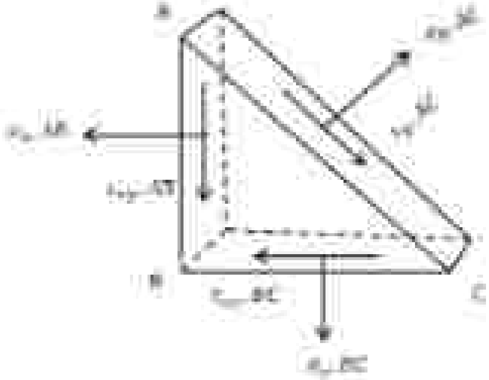
- Though there are different sign conventions, used in different books, yet we shall adopt the following sign conventions, which are widely used and internationally recognized:
  1. All the tensile stresses and strains are taken as positive, whereas all the compressive stresses and strains are taken as negative.
  2. The well-established principles of mechanics is used for the shear stress. The shear stress which tends to rotate the element in the clockwise direction is taken as negative, whereas that which tends to rotate in an anticlockwise direction as positive.
- In the element shown in Figure below, the shear stress on the vertical faces (or x-x axis) is taken as negative, whereas the shear stress on the horizontal faces (or y-y axis) is taken as positive.

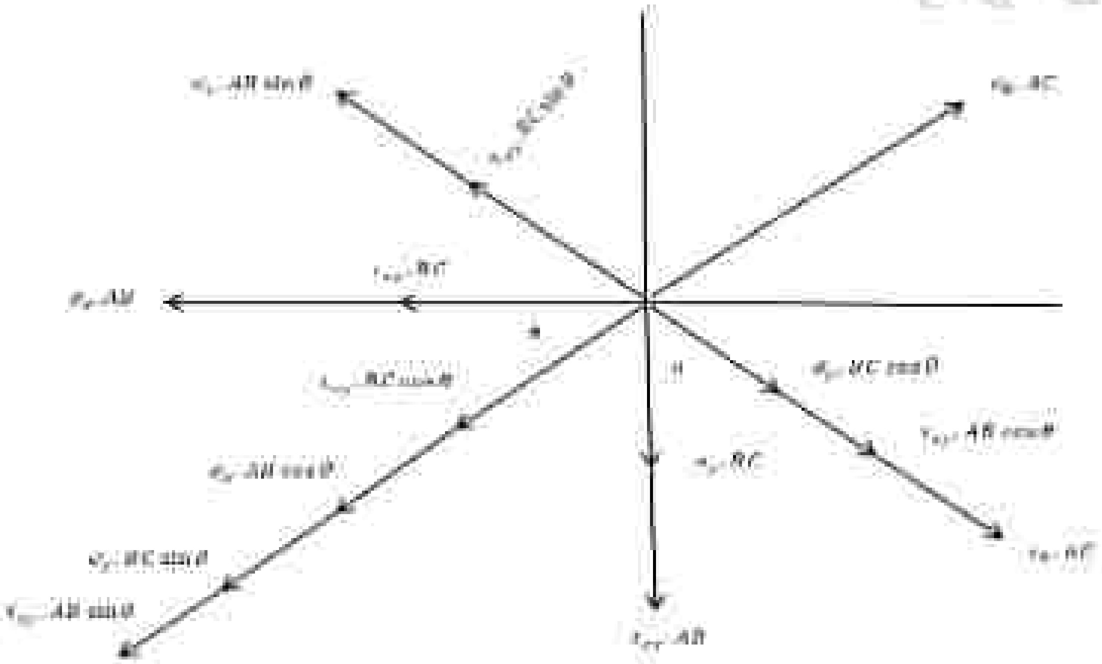


3.1 Determination of normal stress, shear stress and resultant stress on oblique plane

- Consider an elemental rectangular block of unit thickness subjected to two mutually perpendicular stresses σ_x and σ_y and a shear stress τ_{xy} as shown in the figure. We have to calculate the normal stress, shear stress and resultant stress on a plane AC which is inclined at an angle θ with the vertical plane measured in anticlockwise direction.







Resolving the forces acting perpendicular to the inclined plane (i.e. in the direction of σ_θ)

$$F_\theta = \sigma_\theta . AC - \sigma_y . BC \sin\theta - \sigma_x . AB \cos\theta - \tau_{xy} . AB \sin\theta - \tau_{xy} . BC \cos\theta \ldots\ldots\ldots\ldots (i)$$

Resolving the forces acting along the inclined plane (i.e. in the direction of τ_θ)

$$F_\tau = \tau_\theta . AC + \tau_{xy} . AB \cos\theta + \sigma_y . BC \cos\theta - \sigma_x . AB \sin\theta - \tau_{xy} . BC \sin\theta \ldots\ldots\ldots\ldots (ii)$$

Normal stress:

Considering equilibrium of forces acting perpendicular to the inclined plane

$$\sum F_\theta = 0$$

$$\Rightarrow \sigma_\theta . AC = \sigma_y . BC \sin\theta + \sigma_x . AB \cos\theta + \tau_{xy} . AB \sin\theta + \tau_{xy} . BC \cos\theta$$

$$\Rightarrow \sigma_\theta . \frac{AC}{AC} = \sigma_y . \frac{BC}{AC} \sin\theta + \sigma_x . \frac{AB}{AC} \cos\theta + \tau_{xy} . \frac{AB}{AC} \sin\theta + \tau_{xy} . \frac{BC}{AC} \cos\theta$$

$$\Rightarrow \sigma_\theta = \sigma_y . \sin^2\theta + \sigma_x . \cos^2\theta + \tau_{xy} . \cos\theta . \sin\theta + \tau_{xy} . \sin\theta . \cos\theta$$

$$\Rightarrow \sigma_\theta = \sigma_y . \left(\frac{1 - \cos 2\theta}{2}\right) + \sigma_x . \left(\frac{1 + \cos 2\theta}{2}\right) + 2\tau_{xy} . \sin\theta . \cos\theta$$

$$\Rightarrow \sigma_\theta = \frac{\sigma_y}{2} - \frac{\sigma_y \cos 2\theta}{2} + \frac{\sigma_x}{2} + \frac{\sigma_x \cos 2\theta}{2} + \tau_{xy}. \sin 2\theta$$

$$\Rightarrow \sigma_\theta = \left(\frac{\sigma_x + \sigma_y}{2}\right) + \left(\frac{\sigma_x - \sigma_y}{2}\right)\cos 2\theta + \tau_{xy}. \sin 2\theta$$

Shear stress:

Considering equilibrium of forces acting along the inclined plane

$$\sum F_\theta = 0$$

$$\Rightarrow \tau_\theta . AC = -\tau_{xy} . AB \cos\theta - \sigma_y . BC \cos\theta + \sigma_x . AB \sin\theta + \tau_{xy} . BC \sin\theta$$

$$\Rightarrow \tau_\theta . \frac{AC}{AC} = -\tau_{xy} . \frac{AB}{AC}\cos\theta - \sigma_y . \frac{BC}{AC}\cos\theta + \sigma_x . \frac{AB}{AC}\sin\theta + \tau_{xy} . \frac{BC}{AC}\sin\theta$$

$$\Rightarrow \tau_\theta = -\tau_{xy} . \cos^2\theta - \sigma_y . \sin\theta . \cos\theta + \sigma_x . \cos\theta . \sin\theta + \tau_{xy} . \sin^2\theta$$

$$\Rightarrow \tau_\theta = (\sigma_x - \sigma_y)\sin\theta . \cos\theta - \tau_{xy}(\cos^2\theta - \sin^2\theta)$$

$$\Rightarrow \tau_\theta = \left(\frac{\sigma_x - \sigma_y}{2}\right) 2 . \sin\theta . \cos\theta - \tau_{xy}(\cos^2\theta - \sin^2\theta)$$

$$\Rightarrow \tau_\theta = \left(\frac{\sigma_x - \sigma_y}{2}\right)\sin 2\theta - \tau_{xy}\cos 2\theta$$

Resultant stress:

- Magnitude of resultant stress

  $\sigma_R = \sqrt{\sigma_\theta^2 + \tau_\theta^2}$

- Direction of resultant stress

  $\tan\theta = \frac{\tau_\theta}{\sigma_\theta}$

Obliquity:

- The angle between the resultant stress and normal to oblique plane is known as obliquity.
- It is denoted by φ.
- Mathematically,

$$\varphi = \tan^{-1}\frac{\tau_\theta}{\sigma_\theta}$$

For uniaxial loading ($\sigma_y = 0$, $\tau_{xy} = 0$):

Normal stress:

$$\sigma_\theta = \frac{\sigma_x}{2} + \frac{\sigma_x}{2}\cos 2\theta$$

$$\Rightarrow \sigma_\theta = \frac{\sigma_x}{2}(1 + \cos 2\theta)$$

$$\Rightarrow \sigma_\theta = \frac{\sigma_x}{2}(1 + 2\cos^2\theta - 1)$$

$$\Rightarrow \sigma_\theta = \sigma_x \cos^2\theta$$

Shear stress:

$$\tau_\theta = \frac{\sigma_x}{2}\sin 2\theta$$

For biaxial loading ($\tau_{xy} = 0$):

Normal stress:

$$\sigma_\theta = \left(\frac{\sigma_x + \sigma_y}{2}\right) + \left(\frac{\sigma_x - \sigma_y}{2}\right)\cos 2\theta$$

Shear stress:

$$\tau_\theta = \left(\frac{\sigma_x - \sigma_y}{2}\right) sin\, 2\theta$$

For pure shear stress ($\sigma_x = 0, \sigma_y = 0$)

Normal stress:

$$\sigma_\theta = \tau_{xy} \sin 2\theta$$

Shear stress:

$$\tau_\theta = -\tau_{xy} \cos 2\theta$$

3.2 Location of principal plane and computation of principal stress:

Principal planes:

- The planes on which shear stress is zero are known as principal planes. On these planes only normal stress will be acting. These planes are mutually perpendicular.

Principal stress:

- The magnitude of normal stress across the principal plane, is known as principal stress.
- The plane carrying the maximum normal stress is called major principal plane and the corresponding stress is called major principal stress.
- The plane carrying the minimum normal stress is known as minor principal plane and the corresponding stres is called minor principal stress.

Location of principal plane:

Principal planes may be found out by equating the shear stress equation to zero.

$$\tau_\theta = 0$$

$$\Rightarrow \left(\frac{\sigma_x - \sigma_y}{2}\right) sin\, 2\theta - \tau_{xy}\, cos\, 2\theta = 0$$

$$\Rightarrow \left(\frac{\sigma_x - \sigma_y}{2}\right) sin\, 2\theta = \tau_{xy}\, cos\, 2\theta$$

$$\Rightarrow \frac{sin\, 2\theta}{cos\, 2\theta} = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

$$\Rightarrow \tan 2\theta = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

$$\Rightarrow \boldsymbol{\theta\ or\ \theta_P = \frac{1}{2}\tan^{-1}\left(\frac{2\tau_{xy}}{\sigma_x - \sigma_y}\right)}$$

We know that principal planes are two mutually perpendicular planes.

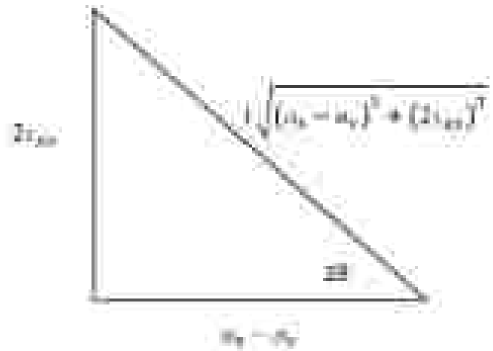
So, if $\boldsymbol{\theta_P = \theta_{P_1}}$

Then, $\boldsymbol{\theta_{P_2} = \theta_{P_1} + 90^0}$

Computation of principal stress:

We know that

$$\tan 2\theta = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$



From the above figure we get,

$$\sin 2\theta = \frac{2\tau_{xy}}{\pm\sqrt{(\sigma_x - \sigma_y)^2 + (2\tau_{xy})^2}}$$

$$\cos 2\theta = \frac{\sigma_x - \sigma_y}{\pm\sqrt{(\sigma_x - \sigma_y)^2 + (2\tau_{xy})^2}}$$

Substituting the above two value in normal stress equation we find the maximum and minimum value of normal stress.

$$\sigma_\theta = \left(\frac{\sigma_x + \sigma_y}{2}\right) + \left(\frac{\sigma_x - \sigma_y}{2}\right)\cos 2\theta + \tau_{xy}.\sin 2\theta$$

$$\Rightarrow \sigma_{1,2} = \left(\frac{\sigma_x + \sigma_y}{2}\right) + \left(\frac{\sigma_x - \sigma_y}{2}\right) \times \frac{\sigma_x - \sigma_y}{\pm\sqrt{(\sigma_x - \sigma_y)^2 + (2\tau_{xy})^2}} + \tau_{xy} \times \frac{2\tau_{xy}}{\pm\sqrt{(\sigma_x - \sigma_y)^2 + (2\tau_{xy})^2}}$$

$$\Rightarrow \sigma_{1,2} = \left(\frac{\sigma_x + \sigma_y}{2}\right) \pm \frac{(\sigma_x - \sigma_y)^2}{2\sqrt{(\sigma_x - \sigma_y)^2 + (2\tau_{xy})^2}} \pm \frac{2(\tau_{xy})^2}{\sqrt{(\sigma_x - \sigma_y)^2 + (2\tau_{xy})^2}}$$

$$\Rightarrow \sigma_{1,2} = \left(\frac{\sigma_x + \sigma_y}{2}\right) \pm \frac{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}{2\sqrt{(\sigma_x - \sigma_y)^2 + (2\tau_{xy})^2}}$$

$$\Rightarrow \sigma_{1,2} = \left(\frac{\sigma_x + \sigma_y}{2}\right) \pm \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}$$

$$\Rightarrow \sigma_{1,2} = \left(\frac{\sigma_x + \sigma_y}{2}\right) \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + (\tau_{xy})^2}$$

Where, σ_1 = Major or maximum principal stress

$$= \left(\frac{\sigma_x + \sigma_y}{2}\right) + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \left(\tau_{xy}\right)^2}$$

σ_2 = Minor or minimum principal stress

$$= \left(\frac{\sigma_x + \sigma_y}{2}\right) - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \left(\tau_{xy}\right)^2}$$

Principal shear stress:

We know that,

$$\tau_\theta = \left(\frac{\sigma_x - \sigma_y}{2}\right) \sin 2\theta - \tau_{xy} \cos 2\theta$$

Differentiate the above equation and equating to zero for finding out the maximum and minimum value of principal shear stress.

$$\frac{d\tau_\theta}{d\theta} = 0$$

$$\Rightarrow \left(\frac{\sigma_x - \sigma_y}{2}\right) . \cos 2\theta \ . 2 + \tau_{xy} \sin 2\theta \ . 2 = 0$$

$$\Rightarrow \tau_{xy} \sin 2\theta \ . 2 = -\left(\frac{\sigma_x - \sigma_y}{2}\right) . \cos 2\theta \ . 2$$

$$\Rightarrow \tan 2\theta = \frac{\sigma_x - \sigma_y}{-2\tau_{xy}}$$

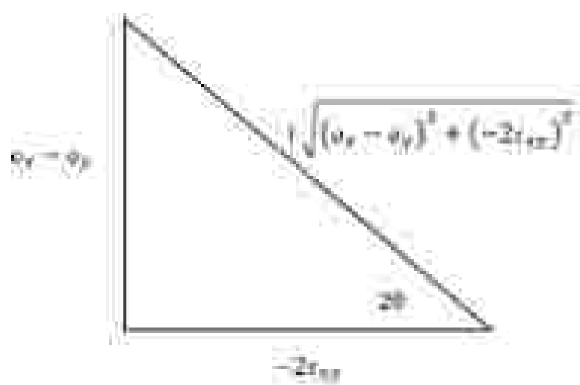
We know that principal shear planes are two mutually perpendicular planes.

So, if $\boldsymbol{\theta_s = \theta_{s_1}}$

Then, $\boldsymbol{\theta_{s_2} = \theta_{s_1} + 90^0}$

We know that,

$$\tan 2\theta = \frac{\sigma_x - \sigma_y}{-2\tau_{xy}}$$



From the above figure we get,

$$\sin 2\theta = \frac{\sigma_x - \sigma_y}{\pm\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}$$

$$\cos 2\theta = \frac{-2\tau_{xy}}{\pm\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}$$

Substituting the above two value in shear stress equation we find the maximum and minimum value of principal shear stress.

$$\tau_\theta = \left(\frac{\sigma_x - \sigma_y}{2}\right)\sin 2\theta - \tau_{xy}\cos 2\theta$$

$$\Rightarrow \tau_{1,2} = \left(\frac{\sigma_x - \sigma_y}{2}\right) \times \frac{\sigma_x - \sigma_y}{\pm\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}} - \tau_{xy} \times \frac{-2\tau_{xy}}{\pm\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}$$

$$\Rightarrow \tau_{1,2} = \frac{(\sigma_x - \sigma_y)^2}{\pm 2\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}} + \frac{2\tau_{xy}^2}{\pm\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}$$

$$\Rightarrow \tau_{1,2} = \pm\frac{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}{2\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}$$

$$\Rightarrow \tau_{1,2} = \pm\frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}$$

$$\Rightarrow \tau_{1,2} = \pm\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + (\tau_{xy})^2}$$

Where, τ_1 = Maximum principal shear stress (+)

τ_2 = Minimum principal shear stress (–)

Example – 1: ***A point in a strained material is subjected to two mutually perpendicular tensile stresses of 200 MPa and 100 MPa. Determine the intensities of normal, shear and resultant stresses on a plane inclined at 30° with the axis of minor tensile stress.***

Solution. Given : Tensile stress along x-x axis (σ_x) = 200 MPa ; Tensile stress along y-y axis (σ_y) = 100 MPa and angle made by plane with the axis of minor tensile stress θ = 30°

Normal stress on the inclined plane

- We know that normal stress on the inclined plane,

$$\sigma_\theta = \left(\frac{\sigma_x + \sigma_y}{2}\right) + \left(\frac{\sigma_x - \sigma_y}{2}\right)\cos 2\theta$$

$$= \left(\frac{200 + 100}{2}\right) + \left(\frac{200 - 100}{2}\right)\cos(2 \times 30^\circ)$$

$$= 150 + (50 \times 0.5) = \mathbf{175\ MPa}$$

Shear stress on the inclined plane

- We know that shear stress on the inclined plane,

$$\tau_\theta = \left(\frac{\sigma_x - \sigma_y}{2}\right) \sin 2\theta$$

$$= \left(\frac{200 - 100}{2}\right) \sin(2 \times 30^0)$$

$$= 50 \times 0.866 = \mathbf{43.3\ MPa}$$

Resultant stress on the inclined plane

- We also know that resultant stress on the inclined plane,

$$\sigma_R = \sqrt{\sigma_\theta^2 + \tau_\theta^2}$$

$$= \sqrt{175^2 + 43.3^2}$$

$$= \mathbf{180.28\ MPa}$$

Example – 2: ***The stresses at point of a machine component are 150 MPa and 50 MPa both tensile. Find the intensities of normal, shear and resultant stresses on a plane inclined at an angle of 55° with the axis of major tensile stress.***

Solution. Given: Tensile stress along x-x axis (σ_x) = 150 MPa ; Tensile stress along y-y axis (σ_y) = 50 MPa and angle made by the plane with the major tensile stress = 55°. So (θ) = 90° – 55° = 35° .

Normal stress on the inclined plane

- We know that the normal stress on the inclined plane,

$$\sigma_\theta = \left(\frac{\sigma_x + \sigma_y}{2}\right) + \left(\frac{\sigma_x - \sigma_y}{2}\right) \cos 2\theta$$

$$= \left(\frac{150 + 50}{2}\right) + \left(\frac{150 - 50}{2}\right) \cos(2 \times 35^0)$$

$$= 100 + (50 \times 0.342) = \mathbf{117.1\ MPa}$$

Shear stress on the inclined plane

- We know that the shear stress on the inclined plane,

$$\tau_\theta = \left(\frac{\sigma_x - \sigma_y}{2}\right) \sin 2\theta$$

$$= \left(\frac{150 - 50}{2}\right) \sin 70^0$$

$$= 50 \times 0.9397 = \mathbf{47\ MPa}$$

Resultant stress on the inclined plane

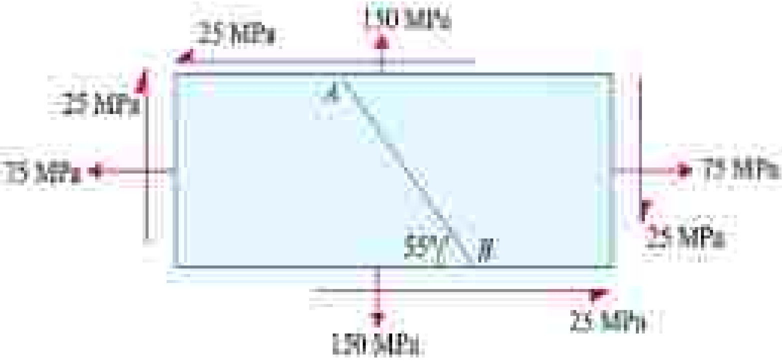
- We know that resultant stress on the inclined plane,

$$\sigma_R = \sqrt{\sigma_\theta^2 + \tau_\theta^2}$$

$$= \sqrt{117.1^2 + 47^2}$$

$$= \mathbf{126.2\ MPa}$$

Example – 3: ***A point in a strained material is subjected to the stresses as shown in Figure. Find the normal and shear stresses on the section AB.***



Solution. Given: Tensile stress along horizontal x-x axis (σ_x) = 75 MPa ; Tensile stress along vertical y-y axis (σ_y) = 150 MPa ; Shear stress (τ_{xy}) = – 25 MPa and angle made by section with the horizontal direction = 55°. So (θ) = 90° – 55° = 35°.

Normal stress on the section AB

- We know that normal stress on the section AB,

$$\sigma_\theta = \left(\frac{\sigma_x + \sigma_y}{2}\right) + \left(\frac{\sigma_x - \sigma_y}{2}\right)\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$= \left(\frac{75 + 150}{2}\right) + \left(\frac{75 - 150}{2}\right)\cos 70^0 - 25 \times \sin 70^0$$

$$= 112.5 + (-37.5 \times 0.342) - 25 \times 0.9397 = \mathbf{76.18\ \mathit{MPa}}$$

Shear stress on the section AB

- We also know that shear stress on the section AB,

$$\tau_\theta = \left(\frac{\sigma_x - \sigma_y}{2}\right)\sin 2\theta - \tau_{xy}\cos 2\theta$$

$$= \left(\frac{75 - 150}{2}\right)\sin 70^0 - (-25 \times \cos 70^0)$$

$$= (-37.5 \times 0.9397) + (25 \times 0.342)$$

$$= -\ 35.24\ +\ 8.55\ = -\ \mathbf{26.69\ \mathit{MPa}}$$

Example – 4: ***A point is subjected to a tensile stress of 250 MPa in the horizontal direction and another tensile stress of 100 MPa in the vertical direction. The point is also subjected to a simple shear stress of 25 MPa, such that when it is associated with the major tensile stress, it tends to rotate the element in the clockwise direction. What is the magnitude of the normal and shear stresses on a section inclined at an angle of 20° with the major tensile stress?***

Solution. Given: Tensile stress in horizontal x-x direction (σ_x) = 250 MPa; Tensile stress in vertical y-y direction (σ_y) = 100 MPa; Shear stress (τ_{xy}) = – 25 MPa and angle made by section with the major tensile stress = 20°. So (θ) = 90° – 20° = 70°.

Magnitude of normal stress

- We know that magnitude of normal stress,

$$\sigma_\theta = \left(\frac{\sigma_x + \sigma_y}{2}\right) + \left(\frac{\sigma_x - \sigma_y}{2}\right)\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$= \left(\frac{250+100}{2}\right) + \left(\frac{250-100}{2}\right)\cos 140^0 + (-25 \times \sin 140^0)$$

$$= 175 + (75 \times -0.766) - (25 \times 0.6428)$$

$$= 175 - 57.45 - 16.07 = \mathbf{101.48\ MPa}$$

Magnitude of shear stress

- We also know that magnitude of shear stress,

$$\tau_\theta = \left(\frac{\sigma_x - \sigma_y}{2}\right)\sin 2\theta - \tau_{xy}\cos 2\theta$$

$$= \left(\frac{250-100}{2}\right)\sin 140^0 - (-25 \times \cos 140^0)$$

$$= (75 \times 0.6428) + (25 \times -0.766)$$

$$= 48.21 - 19.15 = \mathbf{29.06\ MPa}$$

3.3 Location of principal plane and computation of principal stress and Maximum shear stress using Mohr's circle

Graphical Method for the Stresses on an Oblique Section of a Body:

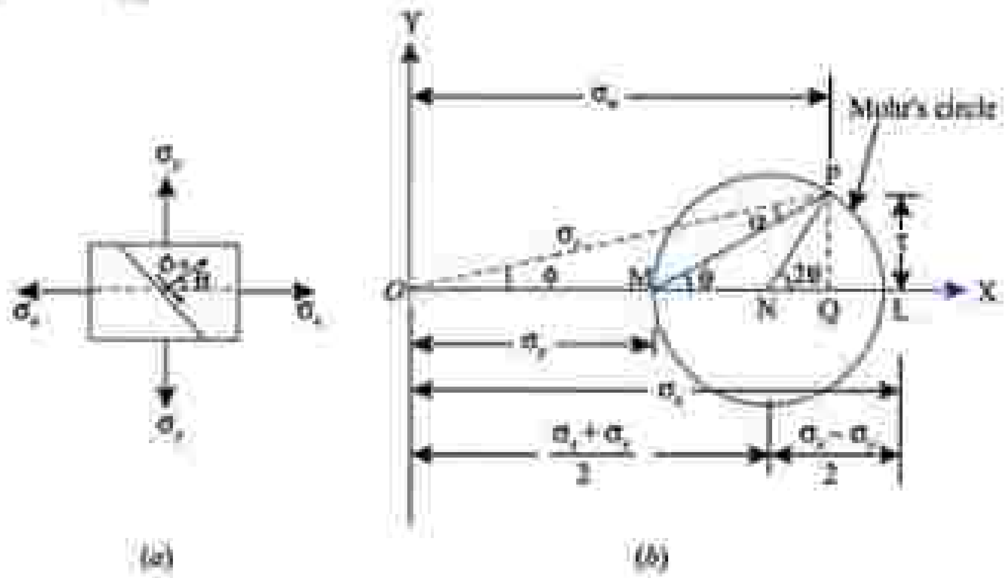
Mohr's Stress Circle:

- The stress components on any inclined plane can easily be found with the help of a geometrical construction known as Mohr's stress circle.

Sign Conventions for Mohr's Stress Circle:

- In order to mark τ in stress system, we will take the clockwise shear as positive and anticlockwise shear as positive.
- Positive values of τ will be above the axis and negative values below the axis.
- If θ is in the anticlockwise direction, the radius vector will be above the axis and θ will be reckoned positive, if θ is in the clockwise direction, it will be negative and the radius vector will be below the axis.
- Tensile stress will be reckoned positive and will be plotted to the right of the origin O. Compressive stress will be reckoned negative and will be plotted to the left of the origin O.

Mohr's circle construction for like stresses:

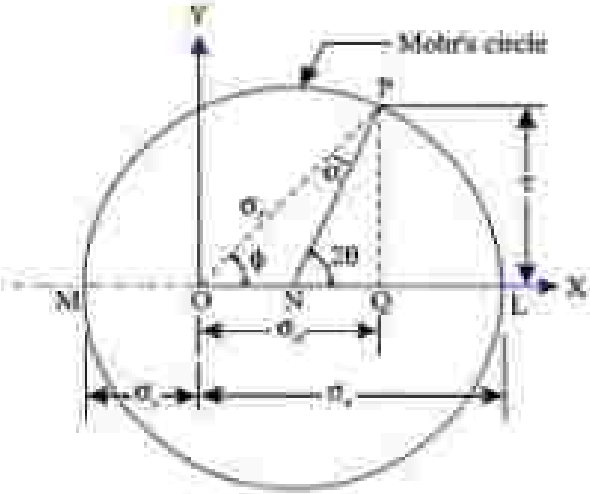


(a) (b)

- Using some suitable scale, measure OL and OM equal to σ_x and σ_y respectively on the axis OX.
- Bisect LM at N.
- With N as Centre and NL or NM as radius, draw a circle.
- At the Centre N draw a line NP at an angle 2θ, in the same direction as the normal to the plane makes with the direction of σ_x. In figure (a) which represents the stress system, the normal to the plane makes an angle θ with the direction of σ_x in the anticlockwise direction. The line NP therefore, is drawn in the anticlockwise direction.
- From P, drop a perpendicular PQ on the axis OX. PQ will represent τ and OQ will represent σ_n.

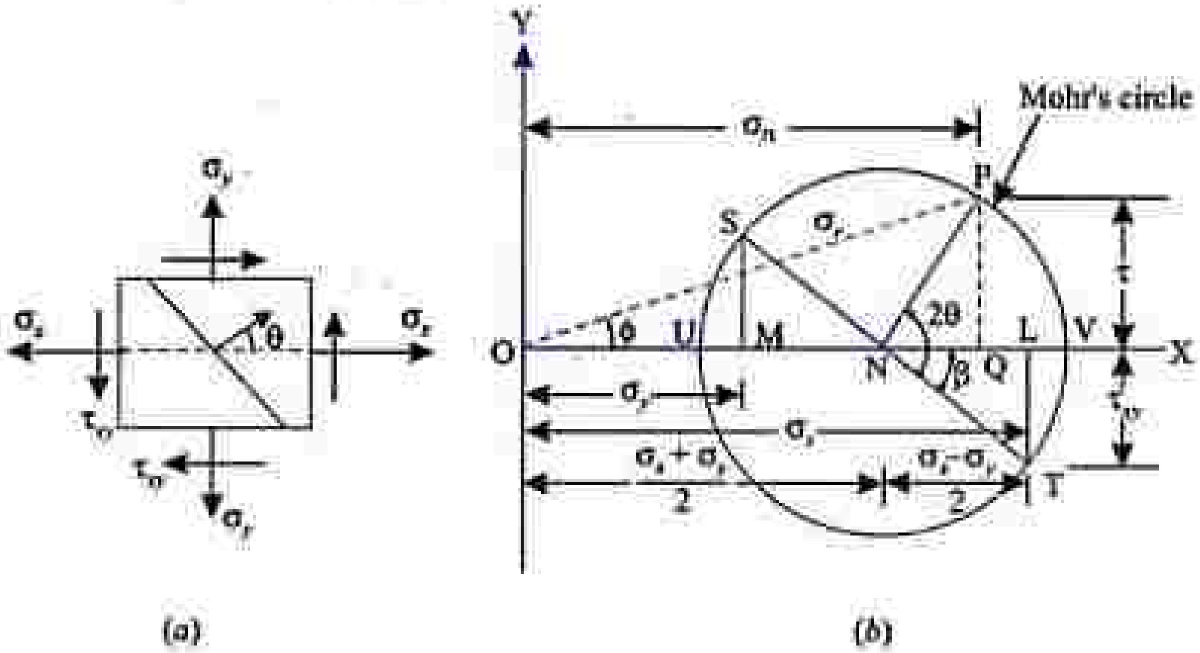
Mohr's circle construction for unlike stresses:

- In σ_x and σ_y case are not like, the same procedure will be followed except that σ_x and σ_y will be measured to the opposite sides of the origin. The construction is given in figure. It may be noted that the direction of σ_n will depend upon its position with respect to the point O. If it is to the right of O, the direction of σ_n will be the same as that of σ_x.



Mohr's circle construction for two perpendicular direct stresses with state of simple shear:

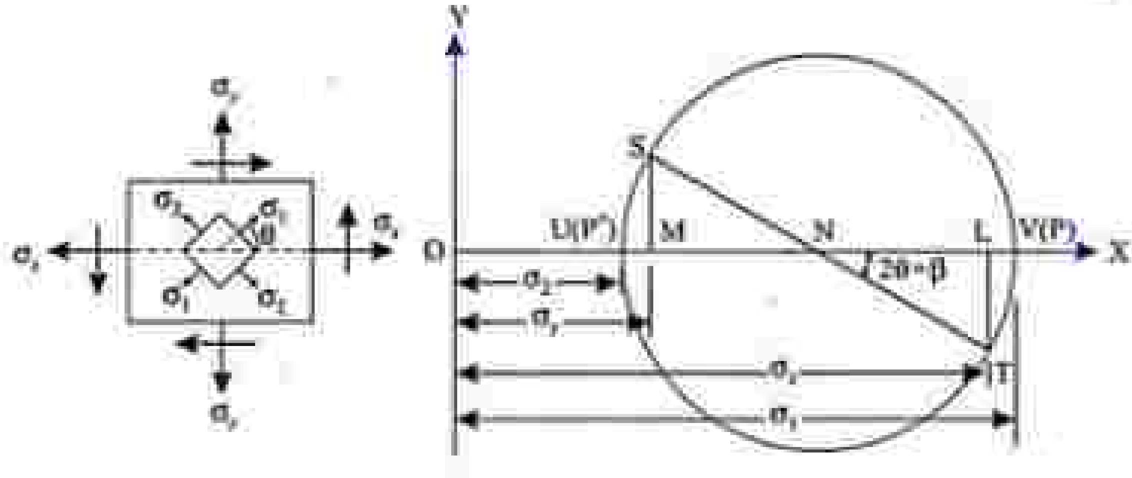
- Following steps of construction are followed if the material is subjected to direct stresses σ_x and σ_y along with a state of simple shear.



(a) (b)

- Using some suitable scale, measure $OL = \sigma_x$ and $OM = \sigma_y$ along the axis OX.
- At L draw LT perpendicular to OX and equal to τ_{xy}. LT has been drawn downward (as per sign convention adopted) because τ_{xy} is acting up with respect to the plane across which σ_x is acting, tending to rotate it in the anticlockwise direction and is negative.
- Similarly, make MS perpendicular to OX and equal to τ_{xy} but above OX.
- Join ST to cut the axis in N.
- With N as Centre and NS or NT as radius, draw a circle.
- At N make NP at angle 2θ with NT in the anticlockwise direction.
- Draw PQ perpendicular to the axis. PQ will give τ while OQ will give σ_n and OP will give σ_r.

Mohr's circle construction for principal stresses:

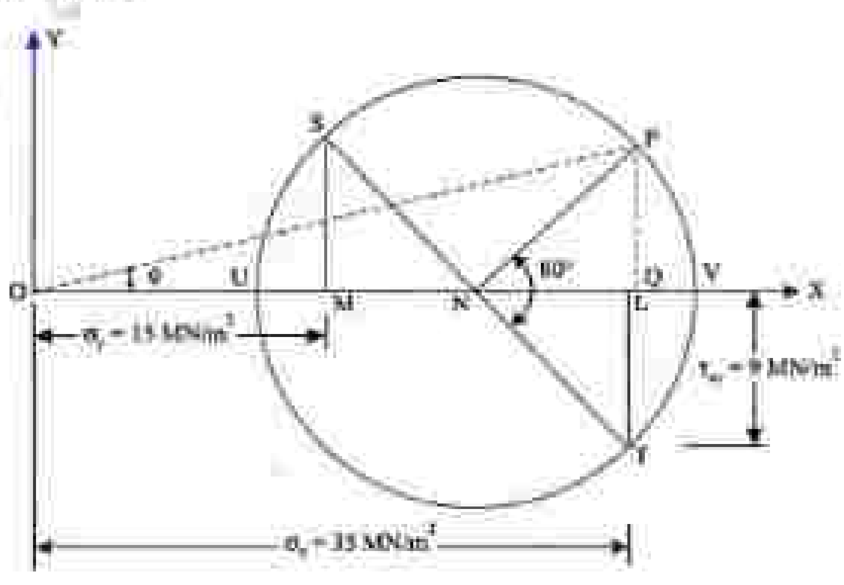


(a) (b)

- Mark OL and OM proportional to σ_x and σ_y.
- At L and M, erect perpendiculars $LT = MS$ proportional to τ_{xy} in appropriate directions.
- Join ST, intersecting the axis in N.
- Since $\tau = 0$, NV represents the major principal plane, P coinciding with V. Similarly, NP' represents minor principal plane, P' coinciding with U.

Example – 5: ***At a point in a bracket the stresses on two mutually perpendicular planes are 35 MN/m² (tensile) and 15 MN/m² (tensile). The shear stress across these planes is 9 MN/m². Find the magnitude and direction of the resultant stress on a plane making an angle of 40° with the plane of first stress. Find also the normal and tangential stresses on the planes.***

Solution: Graphical method:



- Plot OL = 35 MN/m^2 and OM = 15 MN/m^2.
- Drop perpendicular LT and MS, each 9 MN/m^2 as shown in the figure.
- Join ST to get N and draw the Mohr's circle to pass through S and T.
- Draw NP at 80^0 to NT.
- Draw perpendicular to OQ.

From Mohr's circle, we have

- $\sigma_n = OQ$ **= 35.6 MN/m^2** (tensile)
- $\tau = PQ$ **= 8.29 MN/m^2** (shear)
- $\sigma_r = OP$ **= 36.55 MN/m^2**
- ϕ **= 13^0**

POSSIBLE SHORT TYPE QUESTIONS WITH ANSWER:

01. Define principal planes.

Ans: The planes on which shear stress is zero are known as principal planes, on these plines only normal stress will be acting. These planes are mutually perpendicular.

02. Define principal stress and its uses. (W – 2020)

Ans: The magnitude of direct stress, across a principal plane, is known as principal stress. The determination of principal planes and then principal stress is an important factor in the design of various structures and machine components.

3. Define temperature stress.(W-2022)

Ans.When a temperature of a body rises or fall, it will undergo expansion or contraction correspondingly, if free expansion or contraction is allowed. If these changes however are prevented by any means the stresses are produced in the body. These stresses are called temperature stress.

POSSIBLE LONG TYPE QUESTIONS:

01. Define principal planes and principal stresses. Explain their uses. **(W – 2019)**

Hints: refer page no 39

02. A point in a strained material is subjected to two mutually perpendicular tensile stresses of 200 MPa and 100 MPa. Determine the intensities of normal, shear and resultant stresses on a plane inclined at 30^0 with the axis of minor tensile stress. **(W – 2020)**

Hints: refer example no 1

03. At a point in a strained material, the principal stresses are 100 MPa and 50 MPa both tensile. Find the normal and shear stresses at a section inclined at 60^0 with the axis of the major principal stresses. **(W – 2019)**

Hints: refer example no 2

CHAPTER NO. – 04

BENDING MOMENT& SHEAR FORCE

LEARNING OBJECTIVES:

4.1 Types of beam and load
4.2 Concept of Shear force and bending moment
4.3 Shear Force and Bending moment diagram and its salient features illustration in cantilever beam, simply supported beam and over hanging beam under point load and uniformly distributed load

4.1 Types of beam and load

Beam:

Beam is a structural member which is acted upon by a system of external loads at right angles to the axis.

Types of beams:

There are six types of beams.

- Cantilever beam
- Simply supported beam
- Overhanging beam
- Propped cantilever
- Fixed beam
- Continuous beam

Cantilever beam:

- A beam fixed at one end and free (unsupported) at the other end is called a cantilever beam or simply cantilever.

Cantilever Beam

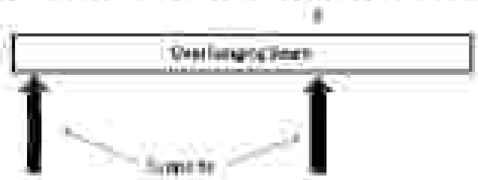
Simply supported beam:

- A beam having its ends freely resting on supports is known as a simply supported beam.

Simply Supported Beam

Overhanging beam:

- A beam having its end portion extended beyond the support is known as overhanging beam.



Propped cantilever:

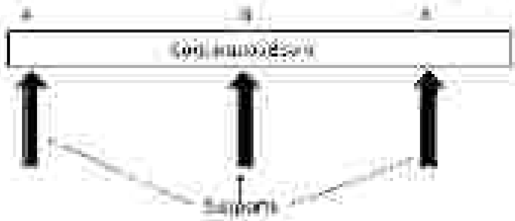
- When a support is provided at some suitable point of a cantilever beam to resist its deflection, then it is known as propped cantilever.

Fixed beam:

- A beam whose both ends are fixed or built-in walls is known as fixed beam.
- It is also called a built-in beam or encastred beam.

Continuous beam:

- When more than two supports are provided for a beam, it is known as continuous beam.
- The supports at the extreme left and right are called end supports and all the other supports are called intermediate supports.



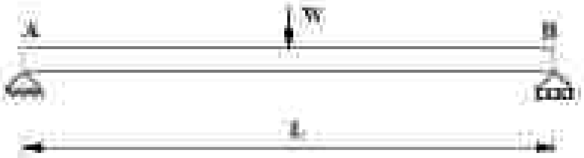
Types of loads:

A beam may be subjected to either or in combination of the following types of loads:

1. Concentrated or point load,
2. Uniformly distributed load and
3. Uniformly varying load

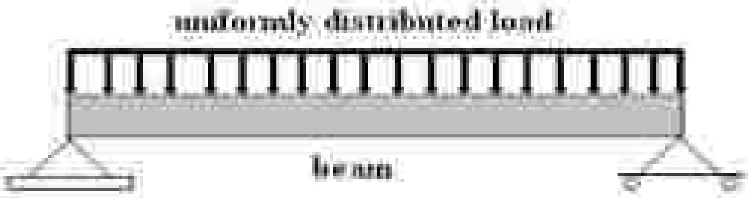
Concentrated or point load:

- It is assumed to act at a point. Practically it is applied over a small area.



Uniformly distributed load:

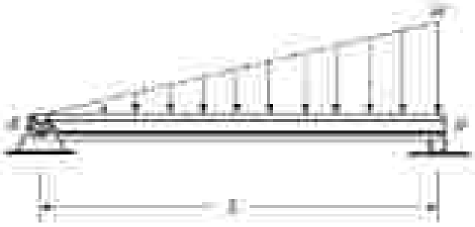
- It is distributed (or spread) uniformly over some length. The intensity of load is constant.



Uniformly varying load:

- It is distributed uniformly over some length of beam but the intensity of load varies.

- The load varies from some value at a position to some other value at another position on the beam in such a way that the change in load per unit length is same over loaded portion of the beam.



4.2 Concepts of Shear force and bending moment

Shear Force:

- The shear force (briefly written as S.F.) at the cross-section of a beam may be defined as the unbalanced vertical force to the right or left of the section.

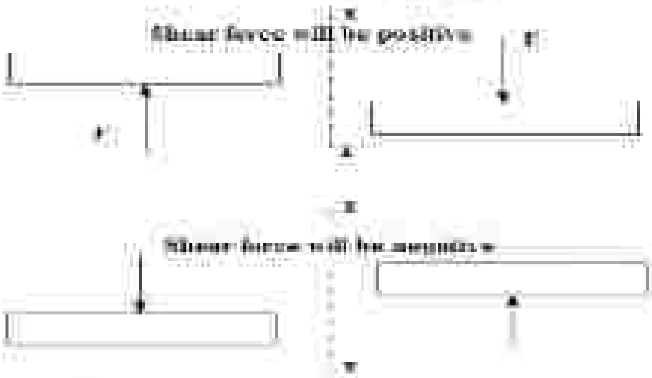
Bending Moment:

- The bending moment (briefly written as B.M.) at the cross-section of a beam may be defined as the algebraic sum of the moments of the forces, to the right or left of the section.

Sign Conventions:

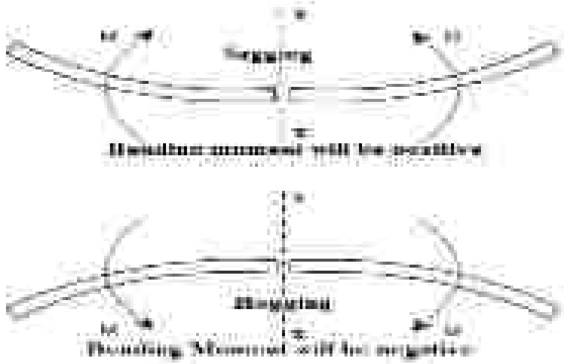
For shear force:

- All the upward forces to the left of the section cause positive shear and those acting downward cause negative shear.
- All the downward forces to the right of the section cause positive shear and those acting upward cause negative shear.



For bending moment:

- When bending moment is calculated using the loads acting to the left side of the section, clockwise moment is positive and anticlockwise moment is negative.
- When bending moment is calculated using the loads acting to the right side of the section, anticlockwise moment is positive and clockwise moment is negative.
- The positive bending moment is often called sagging bending moment and negative as hogging bending moment.



4.3 Shear Force and Bending moment diagram and its salient features illustration in cantilever beam, simply supported beam and over hanging beam under point load and uniformly distributed load

S.F.D: A shear force diagram is one which shows the variation of shear force along the length of the beam.

B.M.D: A bending moment diagram is one which shows the variation of bending moment along the length of the beam.

Note: While drawing the shear force or bending moment diagrams, all the positive values are plotted above the base line and negative values below it.

Relation between Loading, Shear Force and Bending Moment:

The following relations between loading, shear force and bending moment at a point or between any two sections of a beam are important from the subject point of view:

- If there is a point load at a section on the beam, then the shear force suddenly changes (i.e., the shear force line is vertical). But the bending moment remains the same.
- If there is no load between two points, then the shear force does not change (i.e., shear force line is horizontal). But the bending moment changes linearly (i.e., bending moment line is an inclined straight line).
- If there is a uniformly distributed load between two points, then the shear force changes linearly (i.e., shear force line is an inclined straight line). But the bending moment changes according to the parabolic law. (i.e., bending moment line will be a parabola).
- If there is a uniformly varying load between two points then the shear force changes according to the parabolic law (i.e., shear force line will be a parabola). But the bending moment changes according to the cubic law.

Cantilever with a Point Load at its Free End:

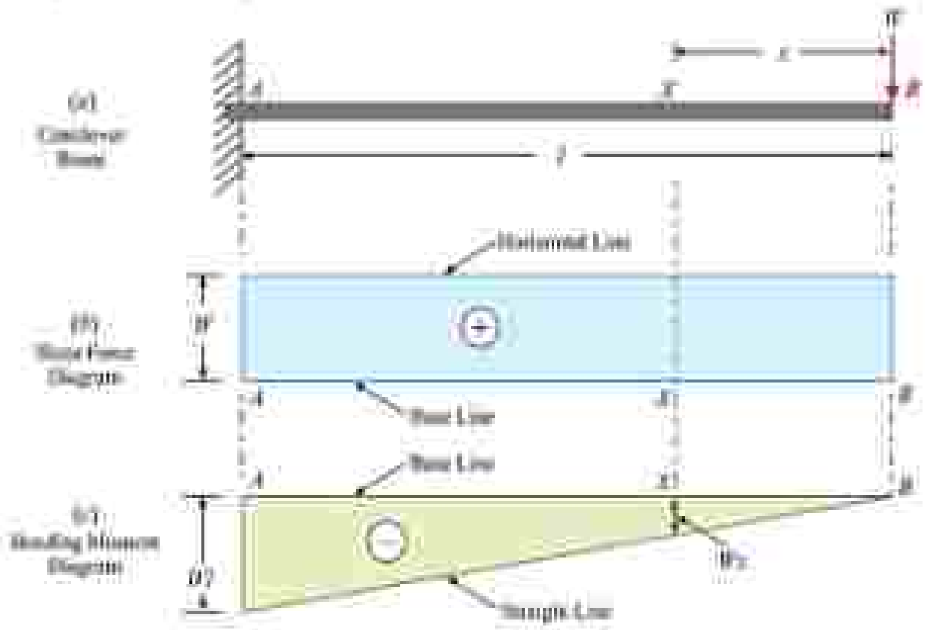
- Consider a cantilever AB of length l and carrying a point load W at its free end B as shown in Figure. We know that shear force at any section X, at a distance x from the free end, is equal to the total unbalanced vertical force. i.e.,

$$F_x = W$$

- and bending moment at this section,

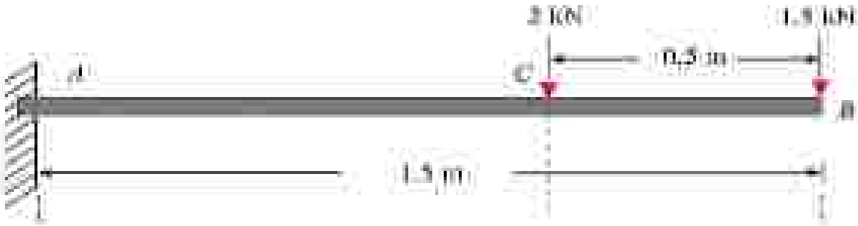
$$M_x = -W.x$$

.....(*Minus sign due to hogging*)



- Thus, from the equation of shear force, we see that the shear force is constant and is equal to W at all sections between B and A. And from the bending moment equation, we see that the bending moment is zero at B (where $x = 0$) and increases by a straight-line law to $-Wl$, at (where $x = l$).

Example – 1: Draw shear force and bending moment diagrams for a cantilever beam of span 1.5 m carrying point loads as shown in Figure.



Solution. Given: Span (l) = 1.5 m; Point load at B (W_1) = 1.5 kN and point load at C (W_2) = 2 kN.

Shear force diagram

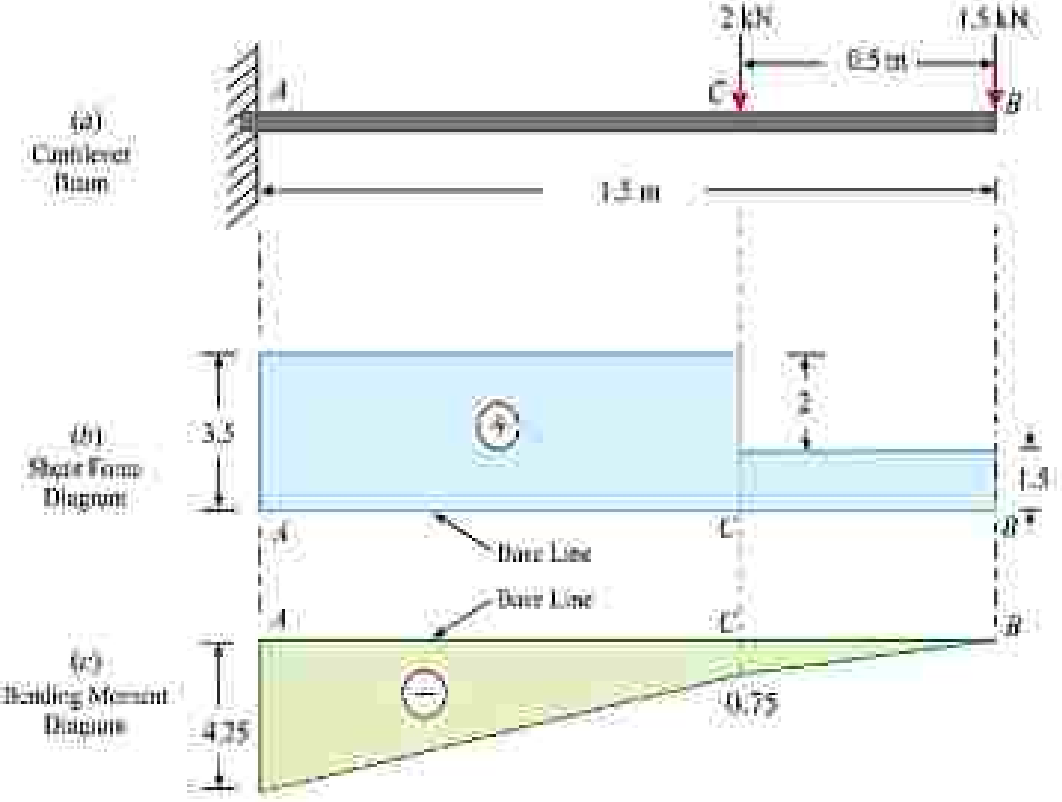
The shear force diagram is shown in Figure (b) and the values are tabulated here:

$$F_B = W_1 = 1.5\ kN$$
$$F_C = (1.5 + W_2) = (1.5 + 2) = 3.5\ kN$$
$$F_A = 3.5\ kN$$

Bending moment diagram

The bending moment diagram is shown in Figure (c) and the values are tabulated here:

$$M_B = 0$$
$$M_C = -[1.5 \times 0.5] = -0.75\ kN.m$$
$$M_A = -[(1.5 \times 1.5) + (2 \times 1)] = -4.25\ kN.m$$



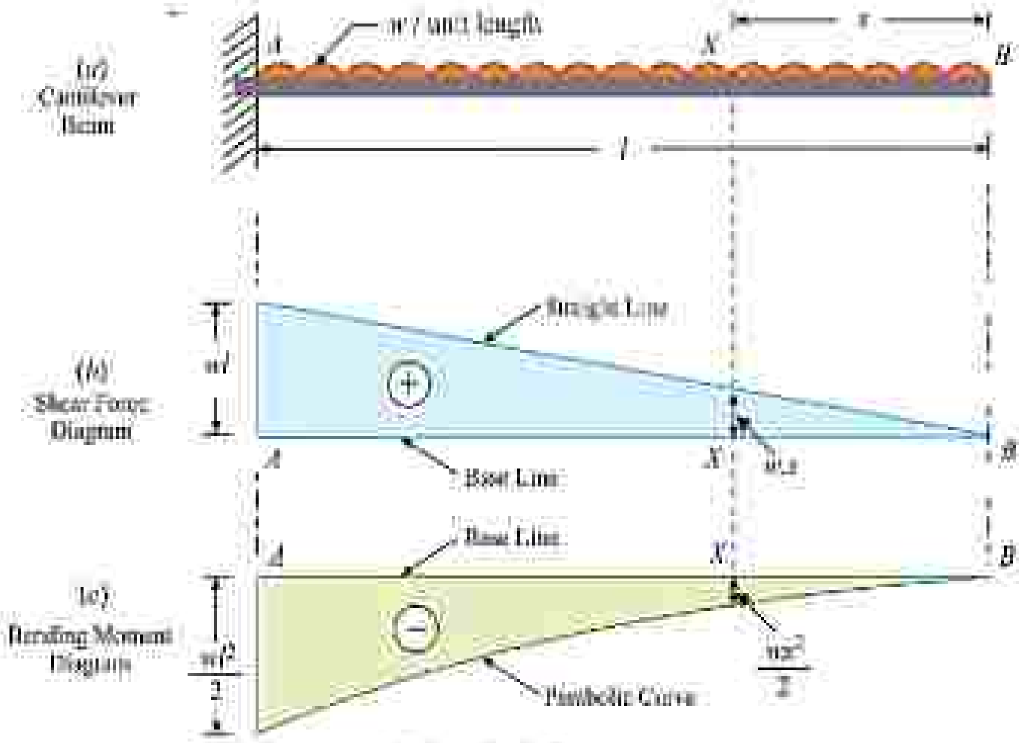
Cantilever with a Uniformly Distributed Load:

Consider a cantilever AB of length l and carrying a uniformly distributed load of w per unit length, over the entire length of the cantilever as shown in Figure.

We know that shear force at any section X, at a distance x from B,

$$F_x = W.x$$

Thus, we see that shear force is zero at B (where x = 0) and increases by a straight-line law to wl at A as shown in Figure.



We also know that bending moment at X,

$$M_x = -wx.\frac{x}{2} = \frac{-wx^2}{2}$$

…… *(Minus sign due to hogging)*

Thus, we also see that the bending moment is zero at B (where $x = 0$) and increases in the form of a parabolic curve to $-\frac{wl^2}{2}$ at B (where $x = l$) as shown in Figure above.

Example – 2: A cantilever beam AB, 2 m long carries a uniformly distributed load of 1.5 kN/m over a length of 1.6 m from the free end. Draw shear force and bending moment diagrams for the beam.

Solution. Given: span (l) = 2 m; Uniformly distributed load (w) = 1.5 kN/m and length of the cantilever CB carrying load (a) = 1.6 m.

Shear force diagram

The shear force diagram is shown in Figure (b) and the values are tabulated here:

$$F_B = 0$$

$$F_C = w.a = 1.5 \times 1.6 = 2.4\ kN$$

$$F_A = 2.4\ kN$$

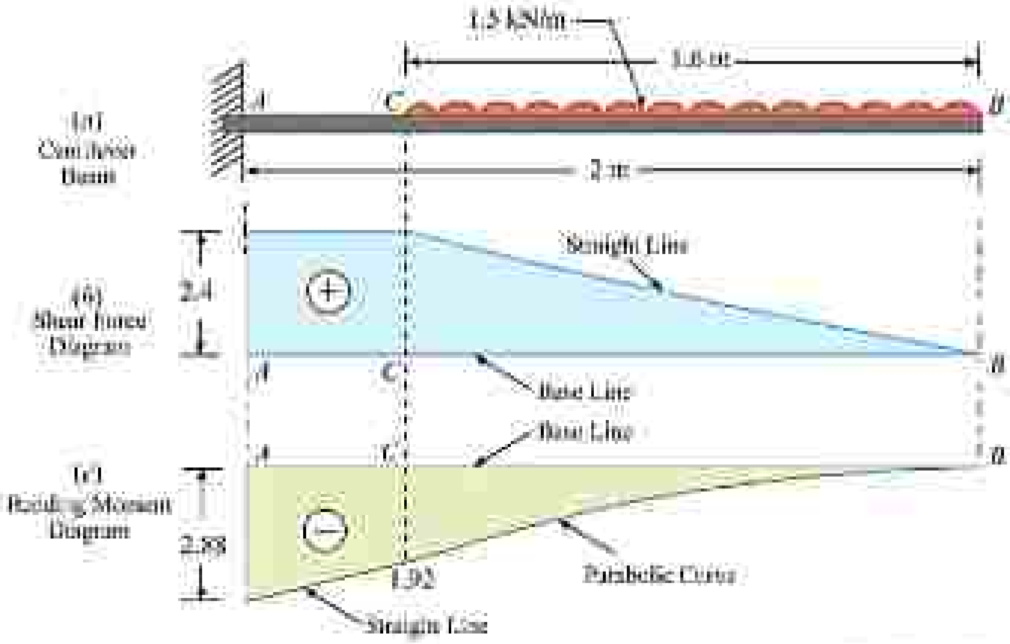
Bending moment diagram

The bending moment diagram is shown in Figure (c) and the values are tabulated here:

$$M_B = 0$$

$$M_C = -\frac{wa^2}{2} = -\frac{1.5 \times (1.6)^2}{2} = -1.92\ kN.m$$

$$M_A = -\left[(1.5 \times 1.6)\left(0.4 + \frac{1.6}{2}\right)\right] = -2.88\ kN.m$$



Simply Supported Beam with a Point Load at its Mid-point:

Consider a simply supported beam AB of span l and carrying a point load W at its mid-point C as shown in the figure. Since the load is at the mid-point of the beam, therefore

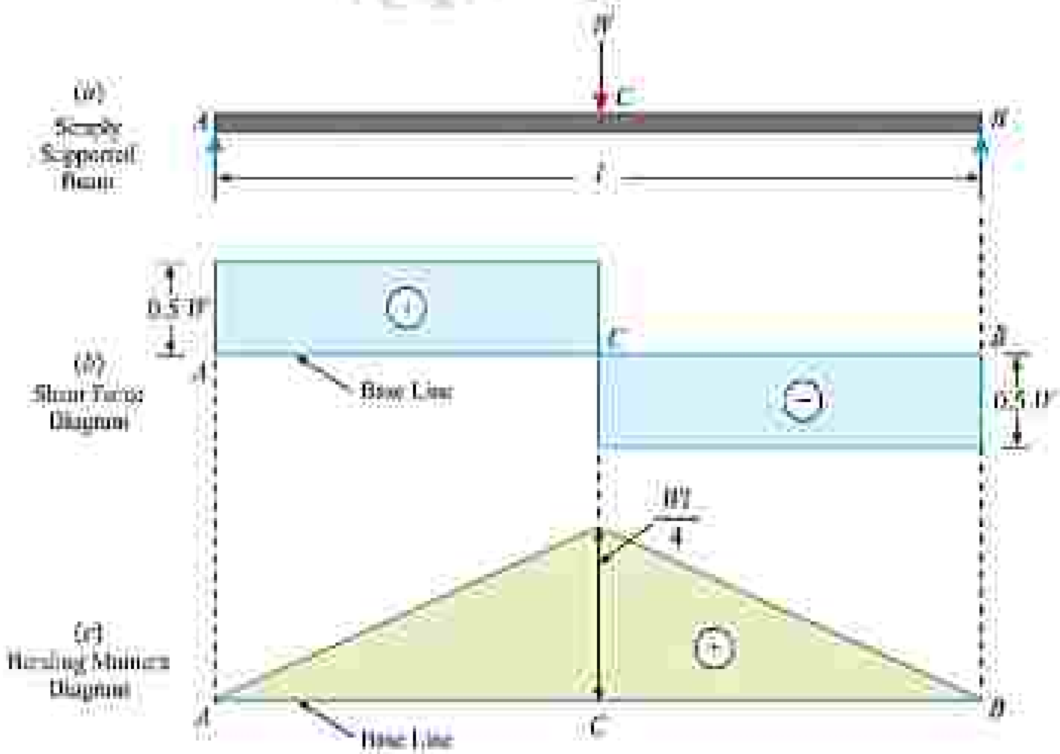
$$R_A + R_B = W$$

Taking moment about A, we get

$$R_B \times l = W \times \frac{l}{2}$$

$$\Rightarrow R_B = \frac{W}{2}$$

$$So\ R_A = \frac{W}{2}$$



(Fig. simply supported beam with a point load)

Thus, we see that the shear force at any section between A and C (i.e., up to the point just before the load W) is constant and is equal to the unbalanced vertical force, i.e., + 0.5 W. Shear force at any section between C and B (i.e., just after the load W) is also constant and is equal to the unbalanced vertical force, i.e., – 0.5 W as shown in Figure (b).

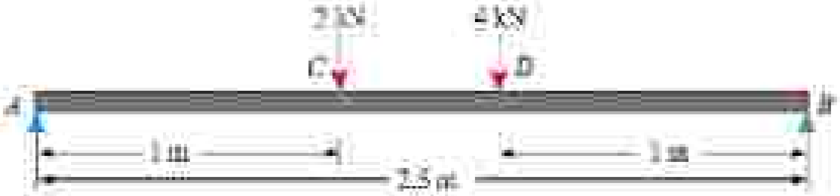
We also see that the bending moment at A and B is zero. It increases by a straight-line law and is maximum at center of beam, where shear force changes sign as shown in Figure (c).

Therefore, bending moment at C,

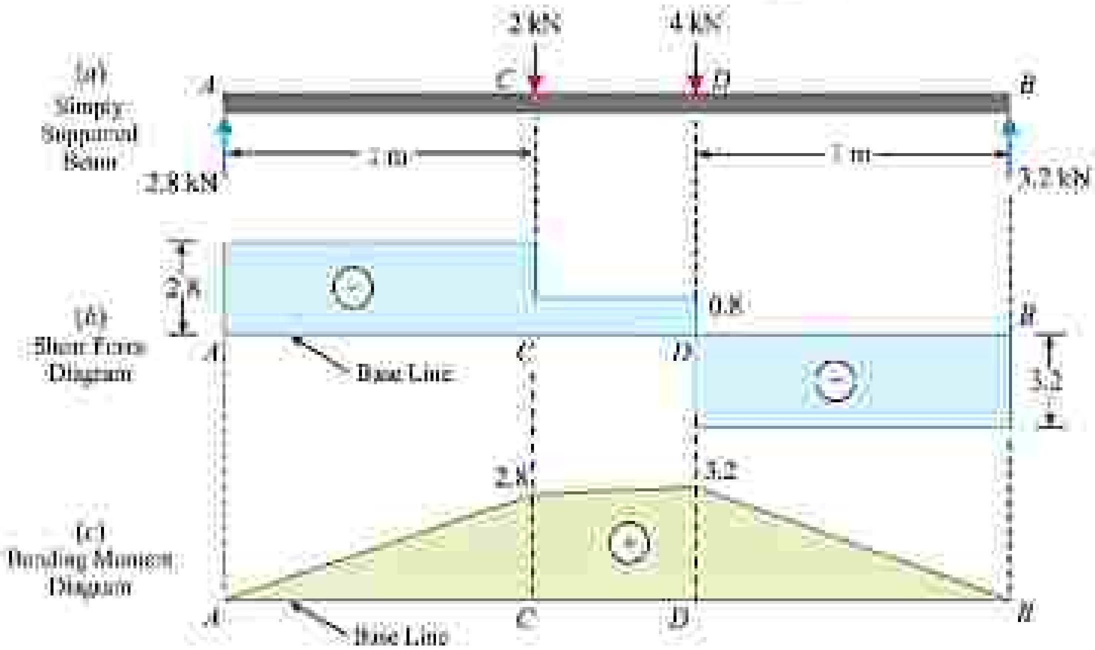
$$M_C = \frac{W}{2} \times \frac{l}{2} = \frac{Wl}{4}$$

Note: B.M. at supports in case of a simply supported beam is always zero.

Example – 3: A simply supported beam AB of span 2.5 m is carrying two point loads as shown in the Figure. Draw the shear force and bending moment diagrams for the beam.



Solution. Given: Span (l) = 2.5 m, Point load at C (W_1) = 2 kN and point load at B (W_2) = 4 kN.



First of all, let us find out the reactions R_A and R_B. Taking moments about A and equating the same,

$$R_B \times 2.5 = (2 \times 1) + (4 \times 1.5) = 8$$

$$R_B = 8/2.5 = 3.2 \text{ kN}$$

$$R_A = (2 + 4) - 3.2 = 2.8 \text{ kN}$$

Shear force diagram

The shear force diagram is shown in Figure (b) and the values are tabulated here:

$$R_A = + R_A = 2.8 \text{ kN}$$

$$F_C = + 2.8 - 2 = 0.8 \text{ kN}$$

$$F_D = 0.8 - 4 = - 3.2 \text{ kN}$$

$$F_B = - 3.2 \text{ kN}$$

Bending moment diagram

The bending moment diagram is shown in Figure (c) and the values are tabulated here

$$M_A = 0$$
$$M_C = 2.8 \times 1 = 2.8 \text{ kN-m}$$
$$M_D = 3.2 \times 1 = 3.2 \text{ kN-m}$$
$$M_B = 0$$

Simply Supported Beam with a Uniformly Distributed Load:

Consider a simply supported beam AB of length l and carrying a uniformly distributed load of w per unit length as shown in Figure. Since the load is uniformly distributed over the entire length of the beam, therefore the reactions at the supports A,

$$R_A = R_B = \frac{wl}{2} = 0.5\ wl$$

We know that shear force at any section X at a distance x from A,

$$F_x = \frac{wl}{2} - wx = w\left(\frac{l}{2} - x\right)$$

At $A, x = 0,\qquad F_A = \frac{wl}{2}$

At $C, x = \frac{l}{2},\qquad F_C = 0$

At $B, x = l,\qquad F_B = -\frac{wl}{2}$

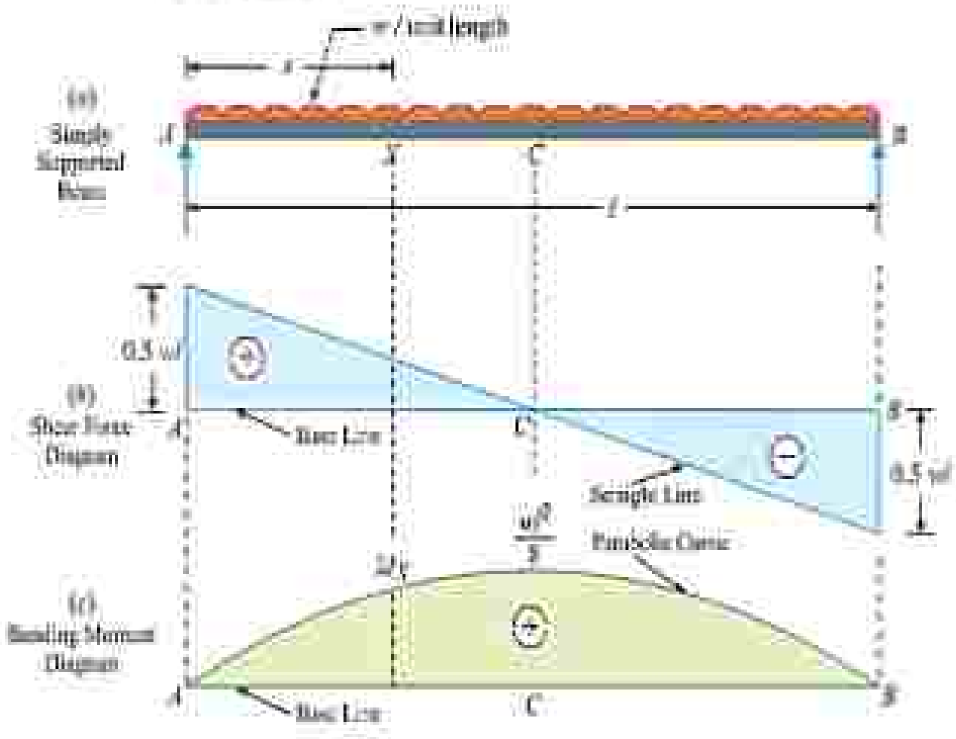
We also know that bending moment at any section at a distance x from A,

$$M_x = \frac{wl}{2}\cdot x - wx\cdot\frac{x}{2} = \frac{w}{2}(lx - x^2)$$

At $A, x = 0,\qquad M_A = 0$

At $B, x = l,\qquad M_B = 0$

At $C, x = \frac{l}{2},\qquad M_C = \frac{w}{2}\left[l \times \frac{l}{2} - \left(\frac{l}{2}\right)^2\right] = \frac{w}{2}\left[\frac{l^2}{2} - \frac{l^2}{4}\right] = \frac{wl^2}{8}$



***Example – 4:** A simply supported beam 6 m long is carrying a uniformly distributed load of 5 kN/m over a length of 3 m from the right end. Draw the S.F. and B.M. diagrams for the beam and also calculate the maximum B.M. on the section.*

Solution. Given: Span (l) = 6 m; Uniformly distributed load (w) = 5 kN/m and length of the beam CB carrying load (a) = 3 m.

First of all, let us find out the reactions R_A and R_B. Taking moments about A and equating the same,

$$R_B \times 6 = (5 \times 3) \times 4.5 = 67.5$$

$$\therefore R_B = 67.5/6 = 11.25 \text{ kN and}$$

$$R_A = (5 \times 3) - 11.25 = 3.75 \text{ kN}$$

Shear force diagram

The shear force diagram is shown in Figure (b) and the values are tabulated here:

$$F_A = + R_A = + 3.75 \text{ kN}$$

$$F_C = + 3.75 \text{ kN}$$

$$F_B = + 3.75 - (5 \times 3) = - 11.25 \text{ kN}$$

Bending moment diagram

The bending moment is shown in Figure (c) and the values are tabulated here:

$$M_A = 0$$

$$M_C = 3.75 \times 3 = 11.25 \text{ kN-m}$$

$$M_B = 0$$

We know that the maximum bending moment will occur at M, where the shear force changes sign. Let x be the distance between C and M. From the geometry of the figure between C and B, we find that

$$\frac{x}{3.75} = \frac{3 - x}{11.25}$$

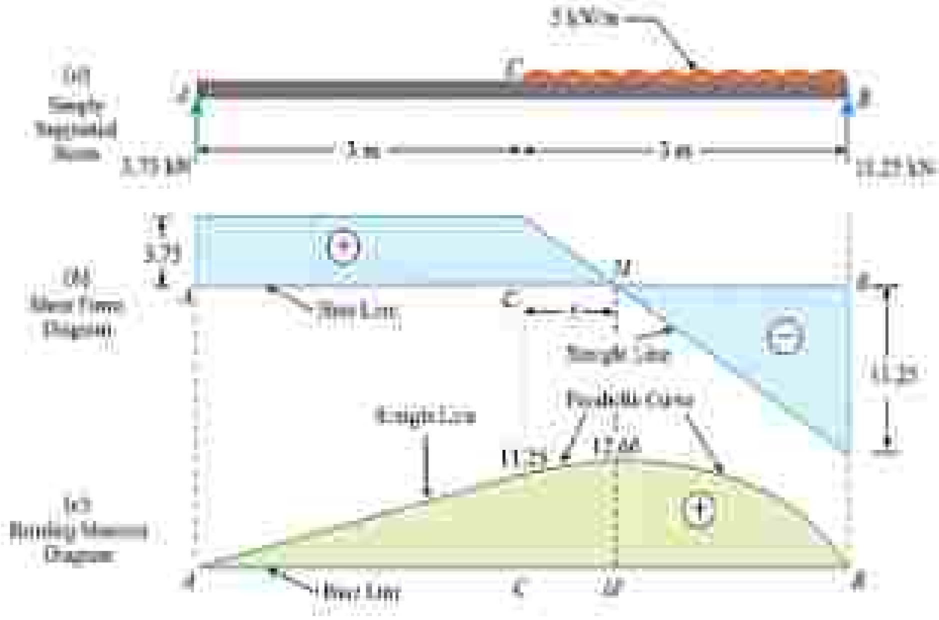
$$\Rightarrow 11.25\,x = 11.25 - 3.75x$$

$$\Rightarrow 15x = 11.25$$

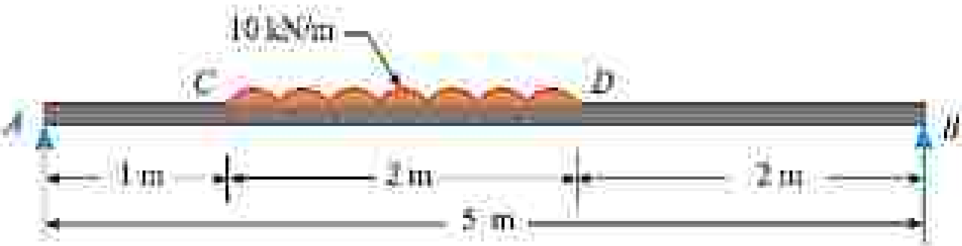
$$\Rightarrow x = \frac{11.25}{15} = 0.75 \text{ m}$$

So maximum bending moment at M,

$$M_M = 3.75 \times (3 + 0.75) - 5 \times 0.75 \times \frac{0.75}{2} = 12.66 \text{ kN.m}$$



Example – 5: A simply supported beam 5 m long is loaded with a uniformly distributed load of 10 kN/m over a length of 2 m as shown in Figure. Draw shear force and bending moment diagrams for the beam indicating the value of maximum bending moment.



Solution. Given: Span (l) = 5 m; Uniformly distributed load (w) = 10 kN/m and length of the beam CD carrying load (a) = 2 m.
First of all, let us find out the reactions R_A and R_B. Taking moments about A and equating the same,

$$R_B \times 5 = (10 \times 2) \times 2 = 40$$

$$\Rightarrow R_B = 40/5 = 8 \text{ kN}$$

$$\text{So, } R_A = (10 \times 2) - 8 = 12 \text{ kN}$$

Shear force diagram
The shear force diagram is shown in Figure (b) and the values are tabulated here:

$$F_A = + RA = + 12 \text{ kN}$$

$$F_C = + 12 \text{ kN}$$

$$F_D = + 12 - (10 \times 2) = - 8 \text{ kN}$$

$$F_B = - 8 \text{ kN}$$

Bending moment diagram
The bending moment diagram is shown in Figure (c) and the values are tabulated here:

$$M_A = 0$$

$$M_C = 12 \times 1 = 12 \text{ kN-m}$$

$$M_D = 8 \times 2 = 16 \text{ kN-m}$$

$$M_B = 0$$

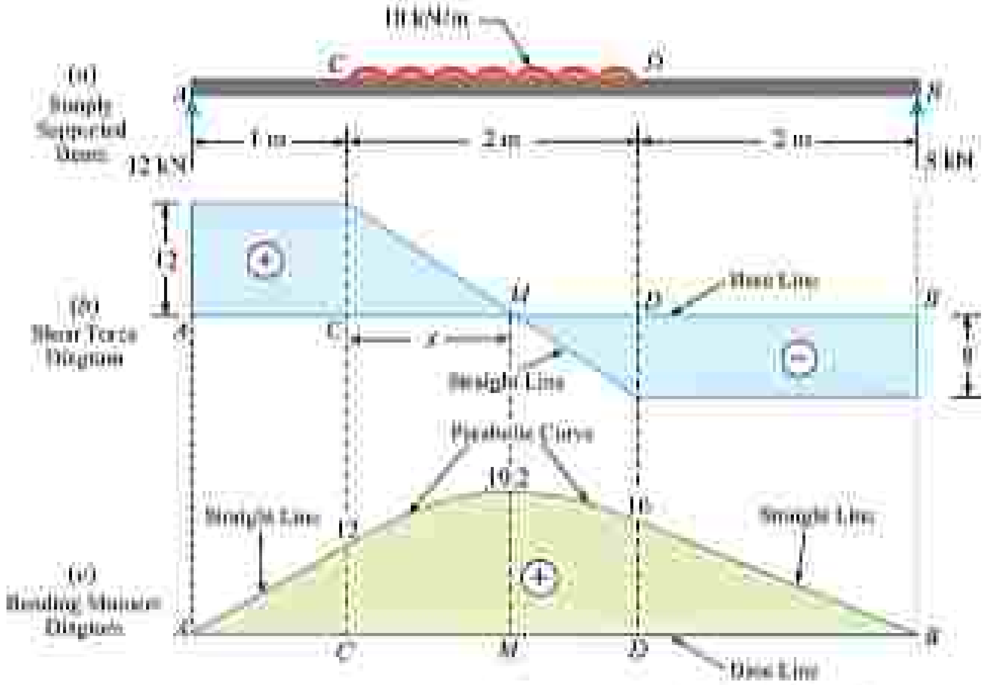
We know that maximum bending moment will occur at M, where the shear force changes sign. Let x be the distance between C and M. From the geometry of the figure between C and D, we find that

$$\frac{x}{12} = \frac{2-x}{8} \quad or \quad 8x = 24 - 12x$$

$$= 20x = 24 \quad or \quad x = 24/20 = 1.2 \, m$$

So maximum bending moment at M,

$$M_M = 12(1+1.2) - \left(10 \times 1.2 \times \frac{1.2}{2}\right) = 19.2 \, kN.m$$



Overhanging Beam:

- It is a simply supported beam which overhangs (i.e., extends in the form of a cantilever) from its support.
- For the purposes of shear force and bending moment diagrams, the overhanging beam is analysed as a combination of a simply supported beam and a cantilever. An overhanging beam may overhang on one side only or on both sides of the supports.

Point of Contraflexure:

- It is the point at which the bending moment changes sign (i.e., from + ve to – ve or vice versa)
- In this point the value of bending moment is zero.
- At this point the beam flexes in opposite direction. This point is also called as point of inflection.

Example – 6: An overhanging beam ABC is loaded as shown in Figure. Draw the shear force and bending moment diagrams and find the point of contraflexure, if any.

Solution. Given: Span (l) = 4 m; Uniformly distributed load (w) = 4.5 kN/m and overhanging length (c)=1m.

First of all, let us find out the reactions R_A and R_B. Taking moment about A and equating the same,

$$R_B \times 3 = (4.5 \times 4) \times 2 = 36$$

$$\Rightarrow R_B = 36/3 = 12 \text{ kN}$$

$$\text{So,} \quad R_A = (4.5 \times 4) - 12 = 6 \text{ kN}$$

Shear force diagram

The shear force diagram is shown in Figure (b) and the values are tabulated here:

$$F_A = + R_A = + 6 \text{ kN}$$

$$F_B = + 6 - (4.5 \times 3) + 12 = 4.5 \text{ kN}$$

$$F_{A-B} = +6 - (4.5 \times 3) = - 7.5 \text{ kN}$$

$$F_C = + 4.5 - (4.5 \times 1) = 0$$

Bending moment diagram

The bending moment diagram is shown in Figure (c) and the values are tabulated here

$$M_A = 0$$

$$M_B = (6 \times 3) - (4.5 \times 3 \times 1.5) = -2.25\ kN.m$$

$$M_C = 0$$

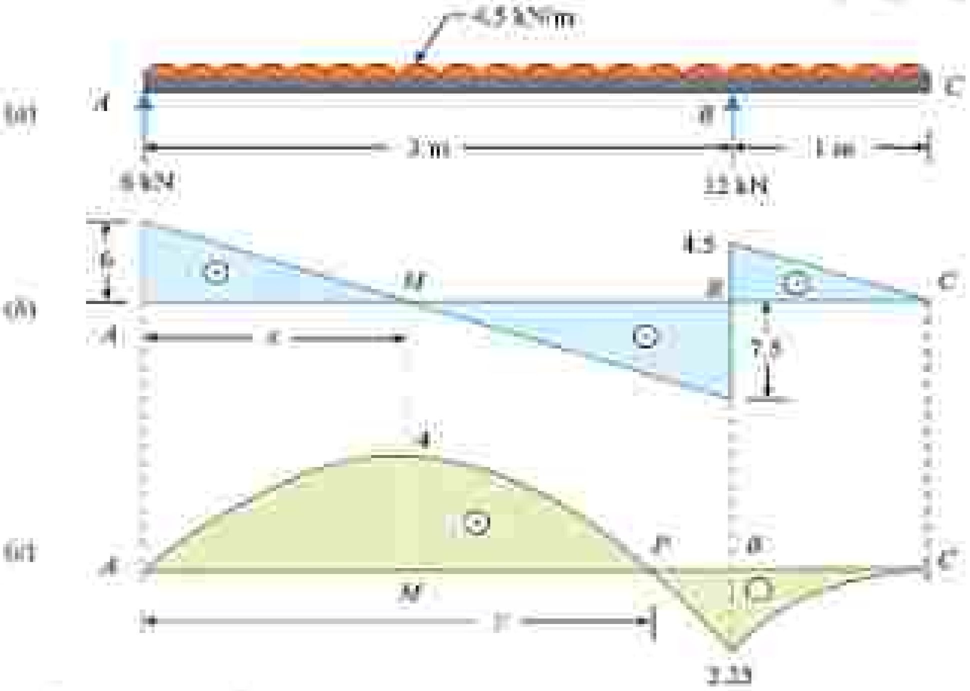
We know that maximum bending moment will occur at M, where the shear force changes sign. Let x be the distance between A and M. From the geometry of the figure between A and B, we find that

$$\frac{x}{6} = \frac{3-x}{7.5} \quad or \quad 7.5x = 18 - 6x$$

$$\Rightarrow 13.5\,x = 18 \quad or \quad x = 18/13.5 = 1.33\ m$$

So maximum bending moment at M,

$$M_M = (6 \times 1.33) - \left(4.5 \times 1.33 \times \frac{1.33}{2}\right) = 4\ kN.m$$



Point of contraflexure

Let P be the point of contraflexure at a distance y from the support A. We know that bending moment at P

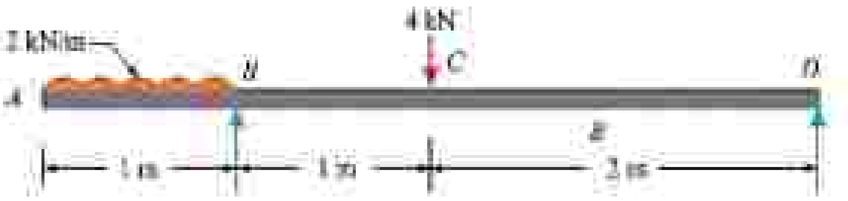
$$M_P = (6 \times y) - \left(4.5 \times y \times \frac{y}{2}\right) = 0$$

$$\Rightarrow 6y - 2.25y^2 = 0$$

$$\Rightarrow 2.25y^2 = 6y$$

$$\Rightarrow y = \frac{6}{2.25} = 2.67\ m$$

Example – 7: A beam ABCD, 4 m long is overhanging by 1 m and carries load as shown in Figure. Draw the shear force and bending moment diagrams for the beam and locate the point of contraflexure.



Solution. Given: Span (l) = 4 m; Uniformly distributed load over AB (w) = 2 kN/m and point load at C (W) = 4 kN.

First of all, let us find out the reactions R_B and R_D. Taking moments about B and equating the same,

$$R_D \times 3 = (4 \times 1) - (2 \times 1) \times 1/2 = 3$$

$$\Rightarrow R_D = 3/3 = 1 \text{ kN}$$

$$\text{So,} \quad R_B = (2 \times 1) + 4 - 1 = 5 \text{ kN}$$

Shear force diagram

The shear force diagram is shown in Figure (b) and the values are tabulated here:

$$F_A = 0$$

$$F_B = 0 - (2 \times 1) + 5 = +3 \text{ kN}$$

$$F_{A-B} = -(2 \times 1) = -2 \text{ kN}$$

$$F_C = +3 - 4 = -1 \text{ kN}$$

$$F_D = -1 \text{ kN}$$

Bending moment diagram

The bending moment diagram is shown in Figure (c) and the values are tabulated here:

$$M_A = 0$$

$$M_B = -(2 \times 1)\, 0.5 = -1 \text{ kN-m}$$

$$M_C = 1 \times 2 = +2 \text{ kN-m}$$

$$M_D = 0$$

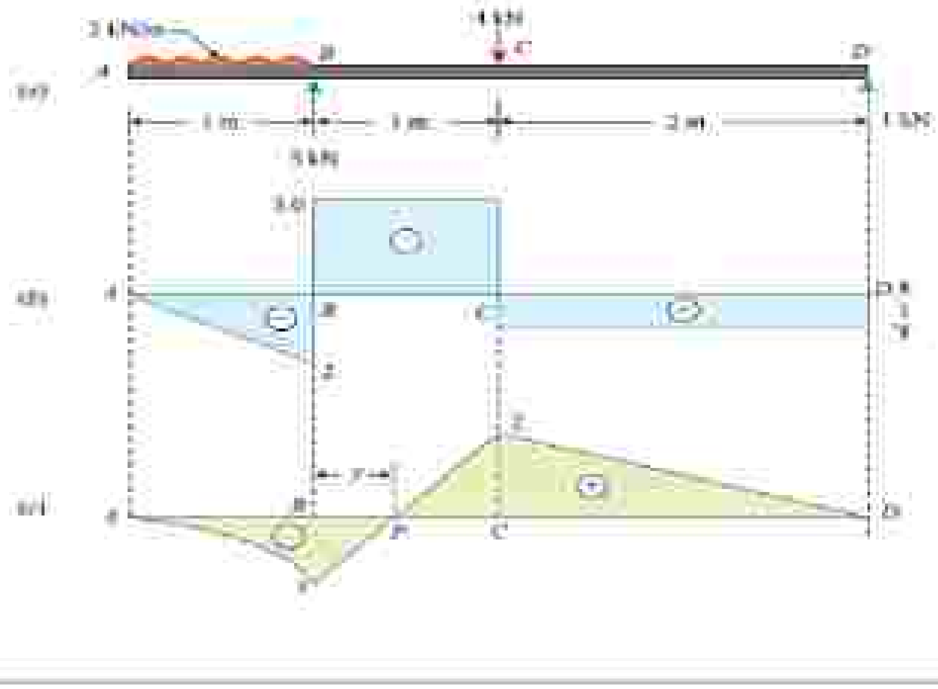
We know that maximum bending moment occurs either at B or C, where the shear force changes sign. From the geometry of the bending moment diagram, we find that maximum negative bending moment occurs at B and maximum positive bending moment occurs at C.

Point of contraflexure

Let P be the point of contraflexure at a distance y from the support B. From the geometry of the figure between B and C, we find that

$$\frac{y}{1} = \frac{1-y}{2} \quad or \quad 2y = 1 - y$$

$$\Rightarrow 3y = 1 \quad or \quad y = 1/3 = 0.33\ m$$



POSSIBLE SHORT TYPE QUESTION WITH ANSWERS:

1. Define shear force and bending moment. (W – 2019)

- The shear force (briefly written as S.F.) at the cross-section of a beam may be defined as the unbalanced vertical force to the right or left of the section.
- The bending moment (briefly written as B.M.) at the cross-section of a beam may be defined as the algebraic sum of the moments of the forces, to the right or left of the section.

2. Define point of contraflexure. (W – 2019, 2020)

- It is the point at which the bending moment changes sign (i.e., from + ve to – ve or vice versa)
- In this point the value of bending moment is zero.
- At this point the beam flexes in opposite direction. This point is also called as point of inflection.

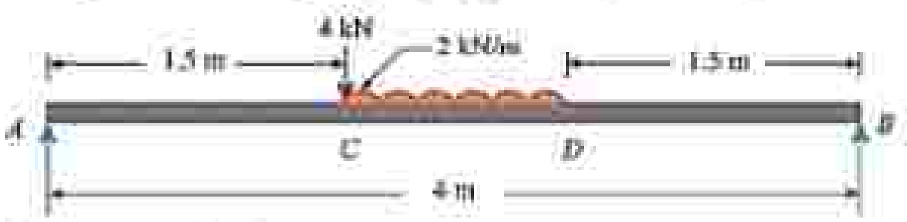
3. Define cantilever beam with example. (W – 2020)

- A beam fixed at one end and free (unsupported) at the other end is called a cantilever beam or simply cantilever.
- Examples: A good example is a balcony, it is supported at one end only, the rest of the beams extends over the open space. Other examples are a cantilever roof in a bus shelter, car park, cantilever bridge (the forth bridge in Scotland is an example of a cantilever truss bridge).

POSSIBLE LONG TYPE QUESTIONS:

1. A simply supported beam of 6 m span carries a point load of 50 kN at a distance of 5 m from its left end. Draw S.F & B.M diagram for the beam. **(W – 2019)**
Hints: Refer examples of S.S.B carrying point loads

2. A simply supported beam of 4 m span is carrying loads as shown in the figure. Draw the shear force and bending moment diagrams for the beam. **(W – 2019)**



Hints: Refer examples of S.S.B carrying point loads and u.d.l

3. Show diagrammatically different types of beams and loads. **(W – 2020)**
Hints: Refer article 4.1

4. A simply supported beam of length 6 m carries point loads of 3 kN and 6 kN at distances of 2 m and 4 m from the left end. Draw the shear force and bending moment diagram for the beam. **(W – 2020).**
Hints: Refer examples of S.S.B carrying point loads

CHAPTER NO. – 05

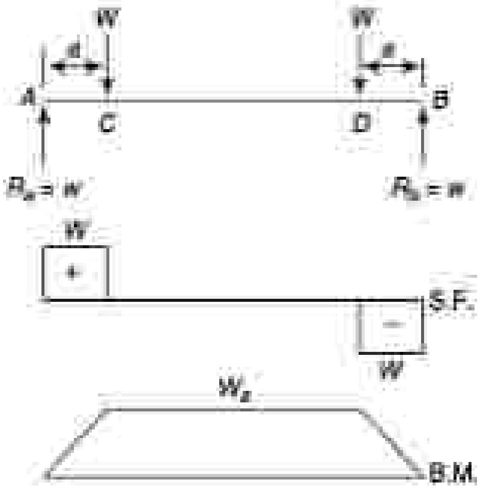
THEORY OF SIMPLE BENDING

LEARNING OBJECTIVES:

5.1 Assumptions in the theory of bending
5.2 Bending equation, Moment of resistance, Section modulus& neutral axis.
5.3 Solve simple problems.

INTRODUCTION:

- The bending moment at a section tends to bend or deflect the beam and the internal stresses resist its bending. The process of bending stops, when every cross-section sets up full resistance to the bending moment. The resistance, offered by the internal stresses, to the bending, is called bending stress, and the relevant theory is called the theory of simple bending.
- **Pure bending or simple bending**: If a member is subjected to equal and opposite couples acting in the same longitudinal plane, the member is said to be in pure bending. A beam or a part of it is said to be in a state of pure bending when it bends under the action of constant bending moment, without any shear force (i.e., S.F = 0 & B.M = constant).

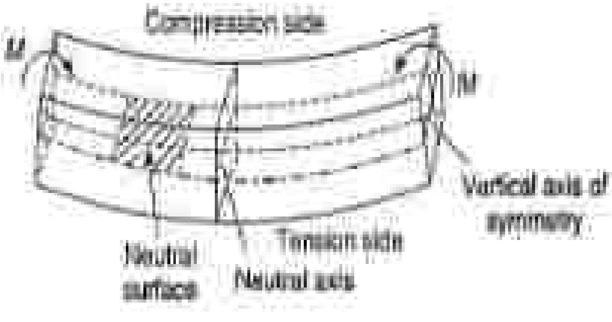


Theory of simple bending:

- Whenever a beam is subjected to simple bending (pure bending) the fibres on one side of beam are subjected to compression while the fibres on the other side are subjected to tension, in between the top and bottom fibres there is a surface where bending stress is zero, this is called neutral surface (neutral layer).

Neutral layer:

- It is a layer in which longitudinal fibres do not change in length. At this layer stress and strain are zero. On one side of layer longitudinal fibres will elongate and on the other side, longitudinal fibres will contract.



Neutral axis:

- It is the line of intersection of neutral layer with the cross section of plane. At neutral axis, stress and strain are zero.

5.1 Assumptions in the Theory of Bending:

The following assumptions are made in the theory of simple bending:

1. The material of the beam is perfectly homogeneous (i.e., of the same kind throughout) and isotropic (i.e., of equal elastic properties in all directions).
2. The beam material is stressed within its elastic limit and thus, obeys Hooke's law.
3. The transverse sections, which were plane before bending, remains plane after bending also.
4. Each layer of the beam is free to expand or contract, independently, of the layer above or below it.
5. The value of E (Young's modulus of elasticity) is the same in tension and compression.
6. The beam is in equilibrium i.e., there is no resultant pull or push in the beam section

5.2 Bending equation, Moment of resistance, Section modulus& neutral axis:

Bending equation:

Consider a small length dx of a beam subjected to a bending moment as shown in Figure (a). As a result of this moment, let this small length of beam bend into an arc of a circle with O as centre as shown in Fig. (b).

Let M = Moment acting at the beam

θ = Angle subtended at the centre by the arc

R = Radius of curvature of the beam.

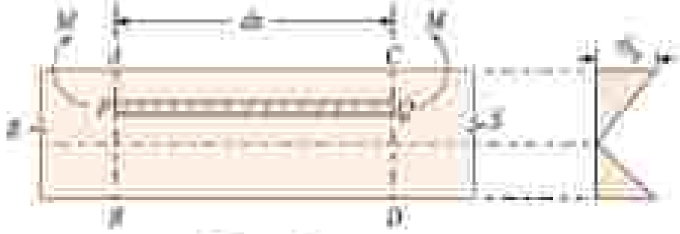


Figure (a)

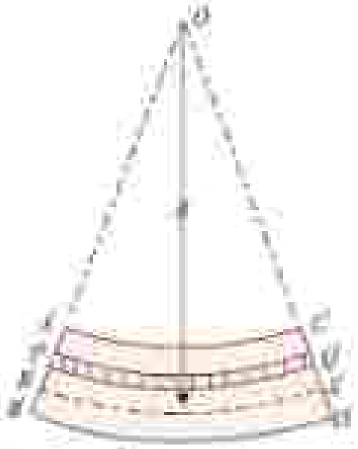


Figure (b)

Now consider a layer PQ at a distance y from RS (the neutral axis of the beam). Let this layer be compressed to $P'Q'$ after bending as shown in Figure (b).

Let $R'S'$ subtend an angle θ at the centre of curvature.

$$\therefore R'S' = R\theta \text{ and } P'Q' = (R - y)\theta$$

Initially the parallel layers would have equal lengths, So that $RS = PQ$ and since there is no stress at the neutral axis, then there is no strain.

So, $RS = R'S' = PQ$

$$Now\ the\ strain\ in\ PQ = \frac{PQ - P'Q'}{PQ}$$

$$\Rightarrow Strain = \frac{R'S' - P'Q'}{PQ}$$

$$\Rightarrow Strain = \frac{R\theta - (R - y)\theta}{R\theta} = \frac{y}{R}$$

Now if the stress in $PQ = \sigma$ and Young's modulus is E, then

$$Strain = \frac{\sigma}{E}$$

$$\therefore \frac{\sigma}{E} = \frac{y}{R} \text{ or } \frac{\sigma}{y} = \frac{E}{R} \ldots\ldots\ldots\ldots\ldots (1)$$

Consider a section of the beam as shown in Figure Let NA be the neutral axis of the section. Now consider a small layer PQ of the beam section at a distance y from the neutral axis as shown in Figure.

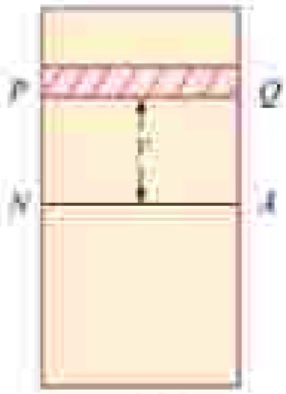
Let, δa = Area of the layer PQ

Then the normal force on this area (δa)

$$= \sigma.\delta a = \frac{E}{R}.y.\delta a$$

Now moment of this force about the neutral axis is

$$= \frac{E}{R} \cdot y . \delta a \times y = \frac{E}{R} \cdot y^2 . \delta a$$



The algebraic sum of all such moments about the neutral axis must be equal to M.

$$Therefore, M = \sum \frac{E}{R} \cdot y^2 . \delta a = \frac{E}{R} \sum y^2 . \delta a$$

The expression $\Sigma y^2 . \delta a$ represents the moment of inertia of the area of the whole section about the neutral axis. Therefore

$$M = \frac{E}{R} \times I \quad or \quad \frac{M}{I} = \frac{E}{R} \dots\dots\dots\dots (ii)$$

So, from equation (i) and (ii) we get,

$$\frac{M}{I} = \frac{\sigma}{y} = \frac{E}{R}$$

Where, M = Moment of resistance

I = Moment of inertia of the section about neutral axis

E = Young's modulus of elasticity

R = Radius of curvature of N.A.

σ = Bending stress

The above equation is known as the '**Bending equation**'.

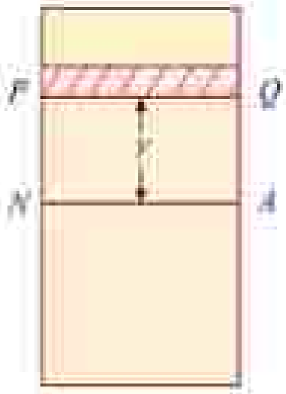
<u>Position of Neutral Axis</u>:

Consider a section of the beam as shown in Figure Let NA be the neutral axis of the section. Now consider a small layer PQ of the beam section at a distance y from the neutral axis as shown in Figure

Let, δa = Area of the layer PQ

Then the normal force on this area (δa)

$$= \sigma . \delta a = \frac{E}{R} . y . \delta a$$



Net normal force on the cross section

$$= \frac{E}{R} \sum y . \delta a$$

For pure bending, net normal force on the cross section = 0

$$\Rightarrow \frac{E}{R} \sum y . \delta a = 0$$

$$\Rightarrow \sum y . \delta a = 0$$

Now $\sum y . \delta a$ is the moment of the sectional area about the neutral axis, and since this moment is zero, the axis must pass through the centre of area.

Hence the neutral axis or neutral layer, passes through the centre of area or centroid.

<u>Section modulus</u>:

- From bending equation, we have

$$\frac{M}{I} = \frac{\sigma}{y}$$

$$\Rightarrow \sigma = \frac{My}{I} = \frac{M}{I/y}$$

$$\Rightarrow \sigma = \frac{M}{Z}$$

$$Where, Z = Section\ modulus = \frac{I}{y}$$

Definition of Z:

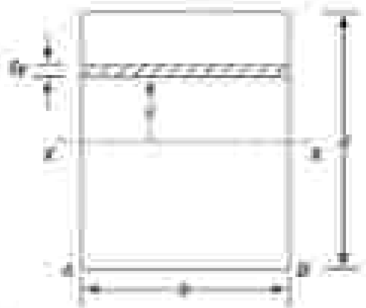
- It is the ratio of moment of inertia of the beam cross section about the neutral axis (I) to the distance of farthest point(y_{max}) of the section from the neutral axis.

$$Z = \frac{I}{y_{max}}$$

Section moduli of Rectangular & circular section:

Rectangular section:

- We know that moment of inertia of a rectangular section about an axis through its centre of gravity,

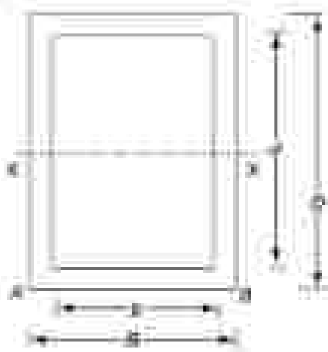


$$I = \frac{bd^3}{12} \quad and \quad y_{max} = \frac{d}{2}$$

$$\therefore Z = \frac{I}{y_{max}} = \frac{bd^3/12}{d/2} = \frac{bd^2}{6}$$

Hollow rectangular section:

- We know that moment of inertia of a hollow rectangular section about an axis through its centre of gravity,

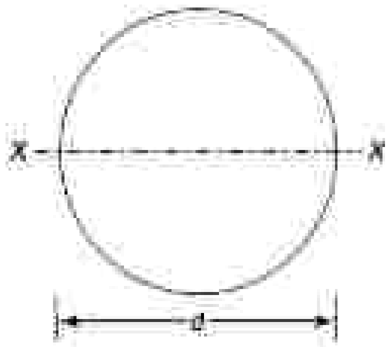


$$I = \frac{BD^3}{12} - \frac{bd^3}{12} = \frac{BD^3 - bd^3}{12} \quad \& \quad y_{max} = \frac{D}{2}$$

$$\therefore Z = \frac{I}{y_{max}} = \frac{(BD^3 - bd^3)/12}{D/2} = \frac{BD^3 - bd^3}{6D}$$

Circular section:

- We know that moment of inertia of a circular section about an axis through its centre of gravity,

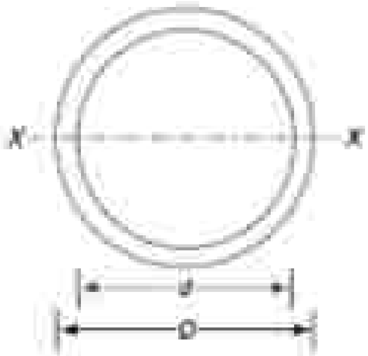


$$I = \frac{\pi d^4}{64} \quad and \quad y_{max} = \frac{d}{2}$$

$$\therefore Z = \frac{I}{y_{max}} = \frac{\pi d^4/64}{d/2} = \frac{\pi d^3}{32}$$

Hollow circular section:

- We know that moment of inertia of a hollow circular section about an axis through its centre of gravity,



$$I = \frac{\pi(D^4 - d^4)}{64} \; and \quad y_{max} = \frac{D}{2}$$

$$\therefore Z = \frac{I}{y_{max}} = \frac{\pi(D^4 - d^4)/64}{D/2} = \frac{\pi}{32}\left(\frac{D^4 - d^4}{D}\right)$$

Moment of resistance:

- The maximum bending moment which can be carried by a given section for a given maximum value of stress is known as the moment of resistance.

Or

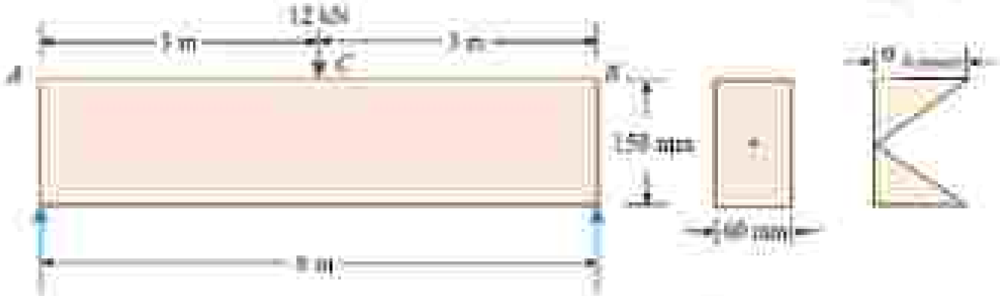
- It is the product of maximum bending stress (σ) and section modulus (Z).
- Mathematically,

$$M = \sigma \times Z.$$

5.3 Solve simple problems:

Example – 1: A rectangular beam 60 mm wide and 150 mm deep is simply supported over a span of 6 m. If the beam is subjected to central point load of 12 kN, find the maximum bending stress induced in the beam section.

Solution. Given: Width (b) = 60 mm; Depth (d) = 150 mm; Span (*l*) = 6 × 10³ mm and load (W) = 12 kN = 12 × 10³ N



We know that maximum bending moment at the centre of a simply supported beam subjected to a central point load,

$$M = \frac{Wl}{4} = \frac{(12 \times 10^3) \times (6 \times 10^3)}{4} = 18 \times 10^6\ N.mm$$

and section modulus of the rectangular section,

$$Z = \frac{bd^2}{6} = \frac{60 \times (150)^2}{6} = 225 \times 10^3\ mm^3$$

Maximum bending stress,

$$\sigma = \frac{M}{Z} = \frac{18 \times 10^6}{225 \times 10^3} = 80\ N/mm^2 = 80\ MPa$$

Example – 2: A rectangular beam 300 mm deep is simply supported over a span of 4 metres. What uniformly distributed load the beam may carry, if the bending stress is not to exceed 120 MPa. Take I = 225 × 10⁶ mm⁴.

Solution. Given: Depth (d) = 300 mm; Span (*l*) = 4 m = 4 × 10³ mm; Maximum bending stress (σ_{max}) = 120 MPa = 120 N/mm² and moment of inertia of the beam section (I) = 225 × 10⁶ mm⁴

Let, w = Uniformly distributed load the beam can carry

We know that distance between the neutral axis of the section and extreme fibre,

$$y = \frac{d}{2} = \frac{300}{2} = 150\ mm$$

and section modulus of the rectangular section,

$$Z = \frac{I}{y} = \frac{225 \times 10^6}{150} = 1.5 \times 10^6\ mm^3$$

Moment of resistance,

$$M = \sigma_{max} \times Z = 120 \times (1.5 \times 10^6) = 180 \times 10^6\ N.mm$$

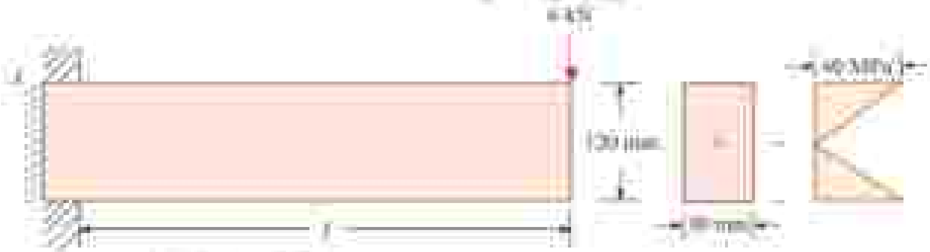
We also know that maximum bending moment at the centre of a simply supported beam subjected to a uniformly distributed load (M),

$$180 \times 10^6 = \frac{wl^2}{8} = \frac{w \times (4 \times 10^3)^2}{8} = 2 \times 10^6\ w$$

$$\Rightarrow w = \frac{180}{2} = 90\ N/mm = 90\ kN/m$$

Example – 3: A cantilever beam is rectangular in section having 80 mm width and 120 mm depth. If the cantilever is subjected to a point load of 6 kN at the free end and the bending stress is not to exceed 40 MPa, find the span of the cantilever beam.

Solution. Given: Width (b) = 80 mm; Depth (d) = 120 mm; Point load (W) = 6 kN= 6×10^3 N and maximum bending stress (σ_{max}) = 40 MPa = 40 N/mm^2.



Let, l = Span of the cantilever beam.

We know that section modulus of the rectangular section,

$$Z = \frac{bd^2}{6} = \frac{80 \times (120)^2}{6} = 192 \times 10^3\ mm^3$$

and maximum bending moment at the fixed end of the cantilever subjected to a point load at the free end,

$$M = Wl = (6 \times 10^3) \times l$$

Maximum bending stress (σ_{max}),

$$40 = \frac{M}{Z} = \frac{(6 \times 10^3) \times l}{192 \times 10^3} = \frac{l}{32}$$

$$\Rightarrow l = 40 \times 32 = 1280\ mm = 1.28\ m$$

Example – 4: A rectangular beam 60 mm wide and 150 mm deep is simply supported over a span of 4 metres. If the beam is subjected to a uniformly distributed load of 4.5 kN/m, find the maximum bending stress induced in the beam.

Solution. Given: Width (b) = 60 mm; Depth (d) = 150 mm; Span (l) = 4 m = 4×10^3 mm and uniformly distributed load (w) = 4.5 kN/m = 4.5 N/mm.

We know that section modulus of the rectangular section,

$$Z = \frac{bd^2}{6} = \frac{60 \times (150)^2}{6} = 225 \times 10^3 mm^3$$

and maximum bending moment at the centre of a simply supported beam subjected to a uniformly distributed load,

$$M = \frac{wl^2}{8} = \frac{4.5 \times (4 \times 10^3)^2}{8} = 9 \times 10^6\ N.mm$$

Maximum bending stress,

$$\sigma_{max} = \frac{M}{Z} = \frac{9 \times 10^6}{225 \times 10^3} = 40\ N/mm^2 = 40\ MPa$$

POSSIBLE SHORT TYPE QUESTION WITH ANSWERS:

01. Define section modulus. (W – 2019, 2020)(W-2022)

- It is the ratio of moment of inertia of the beam cross section about the neutral axis (I) to the distance of farthest point (y_{max}) of the section from the neutral axis.

$$Z = \frac{I}{y_{max}}$$

02. Define moment of resistance.

- The maximum bending moment which can be carried by a given section for a given maximum value of stress is known as the moment of resistance.

Or

- It is the product of maximum bending stress (σ) and section modulus (Z).

- Mathematically,

$$M = \sigma \times Z$$

03. What is pure bending?

- If a member is subjected to equal and opposite couples acting in the same longitudinal plane, the member is said to be in pure bending

4. Define point of contra-flexure.(W-2022)

Ans. Since the over hanging beam is a combination of simple supported beam and cantilever beam so its bending moment diagram has both positive and negative value The point at which on the beam the bending moment changes sign is known as point of contra-flexure.

5.State different types of beam.(W-2021)

Ans:1.cantilever beam
2.Simply supported beam
3.over hanging beam
4.propped cantilever beam
5.fixed beam
6.continuous beam

POSSIBLE LONG TYPE QUESTIONS:

01. State the assumptions made in theory of simple bending. **(W – 2019)**
Hints: Refer article 5.1

02. A beam 3 m long has rectangular section of 80 mm width and 120 mm depth. If the beam is carrying a uniformly distributed load of 10 kN/m, find the maximum bending stress developed in the beam. **(W – 2019)**
Hints: Refer Example - 4

03. Derive the formula of section modulus for rectangular section and circular section. **(W – 2020)**
Hints: Refer (page no 67)

04. Prove the relation

$$\frac{M}{I} = \frac{\sigma}{y} = \frac{E}{R}$$

Where, M = Bending Moment

I = Moment of inertia

E = Young's modulus

R = Radius of curvature

σ = Bending stress in a fiber, at a distance y from the neutral axis **(W – 2020)(W-2022)**

Hints: Refer article 5.2

5.Find the generalized equation for shear force and bending moment of a simply supported beam with uniformly distributed load.(W-2021)

CHAPTER NO. – 06

COMBINED DIRECT & BENDING STRESSES

LEARNING OBJECTIVES:

6.1 Define column
6.2 Axial load, Eccentric load on column
6.3 Direct stresses, Bending stresses, Maximum& Minimum stresses. Numerical problems on above.
6.4 Buckling load computation using Euler's formula (no derivation) in Columns with various end conditions.

6.1 Define column:

- A column is a long vertical slender bar or vertical member, subjected to an axial compressive load and fixed rigidly at both ends.

6.2 Axial load, Eccentric load on column:

Axial load:

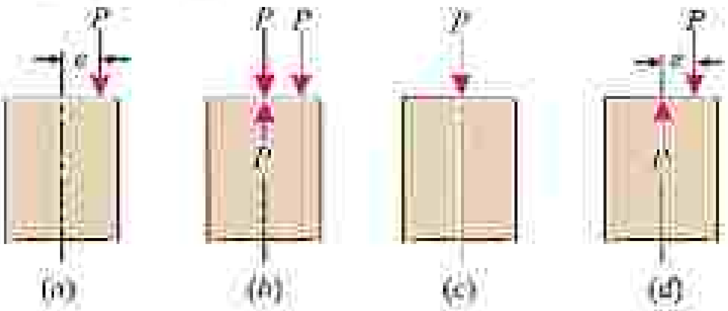
- A load, whose line of action coincide with the axis of a column, is known as an axial load.

Eccentric load

- A load, whose line of action does not coincide with the axis of a column or a strut, is known as an eccentric load.

Example: A bucket full of water, carried by a person in his hand, is an excellent example of an eccentric load. A little consideration will show that the man will feel this load as more severe than the same load, if he had carried the same bucket over his head. The simple reason for the same is that if he carries the bucket in his hand, then in addition to his carrying bucket, he has also to lean or bend on the other side of the bucket, so as to counteract any possibility of his falling towards the bucket. Thus, we say that he is subjected to:
1. Direct load, due to the weight of bucket (including water) and
2. Moment due to eccentricity of the load.

Columns with Eccentric Loading:



Consider a column subjected to an eccentric loading. The eccentric load may be easily analysed as shown in Figure above and as discussed below:

- The given load P, acting at an eccentricity of e, is shown in Fig. (a).
- Let us introduce, along the axis of the strut, two equal and opposite forces P as shown in Fig. (b).
- The forces thus acting, may be split up into three forces.
- One of these forces will be acting along the axis of the strut. This force will cause a direct stress as shown in Fig. (c).
- The other two forces will form a couple as shown in Fig. (d). The moment of this couple will be equal to P × e (This couple will cause a bending stress).

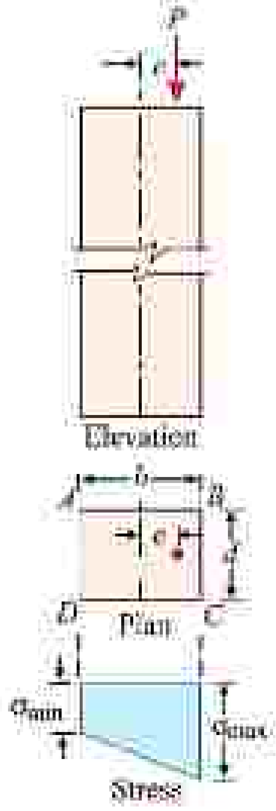
6.3 Direct stresses, Bending stresses, Maximum& Minimum stresses. Numerical problems on above:

Symmetrical Columns with Eccentric Loading about One Axis:

Consider a column ABCD subjected to an eccentric load about one axis (i.e., about y-y axis) as shown in the figure.

Let, P = Load acting on the column,
e = Eccentricity of the load,
b = Width of the column section and
d = Thickness of the column.

Area of column section, $A = b \cdot d$



Moment of inertia of the column section about an axis through its centre of gravity and parallel to the axis about which the load is eccentric (i.e., y-y axis in this case),

$$I = \frac{db^3}{12}$$

and modulus of section,

$$Z = \frac{I}{y} = \frac{db^3/12}{b/2} = \frac{db^2}{6}$$

We know that direct stress on the column due to the load,

$$\sigma_0 = \frac{P}{A}$$

and moment due to load,

$$M = P.e$$

Bending stress at any point of the column section at a distance y from y-y axis,

$$\sigma_b = \frac{M.y}{I} = \frac{M}{Z}$$

Now for the bending stress at the extreme, let us substitute y = b/2 in the above equation,

$$\sigma_b = \frac{M.(b/2)}{db^3/12} = \frac{6M}{db^2} = \frac{6\,P.e}{A.b}$$

We know that an eccentric load causes a direct stress as well as bending stress. It is thus obvious that the total stress at the extreme fibre,

$$\sigma = \sigma_0 \pm \sigma_b = \frac{P}{A} \pm \frac{6\,P.e}{A.b} \ldots (In\ terms\ of\ eccentricity)$$

$$= \frac{P}{A} \pm \frac{M}{Z} \ldots (In\ terms\ of\ modulus\ of\ section)$$

The +ve or –ve sign will depend upon the position of the fibre with respect to the eccentric load. A little consideration will show that the stress will be maximum at the corners B and C (because these corners are near the load), whereas the stress will be minimum at the corners A and D (because these corners are away from the load). The total stress along the width of the column will vary by a straight-line law.

The maximum stress,

$$\sigma_{max} = \frac{P}{A} + \frac{6\,P.e}{A.b} = \frac{P}{A}\left(1 + \frac{6e}{b}\right) \ldots\ldots\ldots (In\ terms\ of\ eccentricity)$$

$$= \frac{P}{A} + \frac{M}{Z} \ldots\ldots\ldots (In\ terms\ of\ modulus\ of\ section)$$

The minimum stress,

$$\sigma_{min} = \frac{P}{A} - \frac{6\,P.e}{A.b} = \frac{P}{A}\left(1 + \frac{6e}{b}\right) \ldots\ldots\ldots (In\ terms\ of\ eccentricity)$$

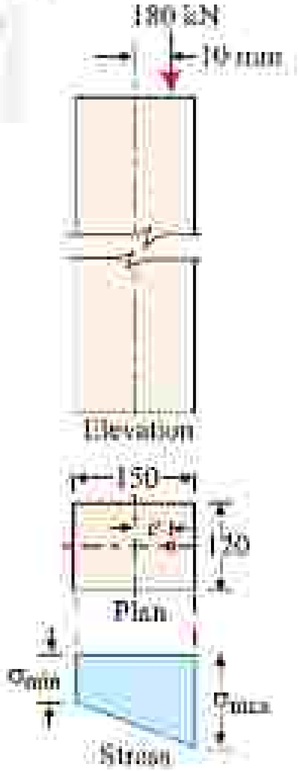
$$= \frac{P}{A} - \frac{M}{Z} \ldots\ldots\ldots\ldots\ldots (In\ terms\ of\ modulus\ of\ section)$$

<u>Notes:</u> From the above equations, we find that

- If σ_0 is greater than σ_b, the stress throughout the section, will be of the same nature (i.e., compressive).
- If σ_0 is equal to σ_b, even then the stress throughout the section will be of the same nature. The minimum stress will be equal to zero, whereas the maximum stress will be equal to $2 \times \sigma_0$.
- If σ_0 is less than σ_b, then the stress will change its sign (partly compressive and partly tensile).

Example – 1: A rectangular strut is 150 mm and 120 mm thick. It carries a load of 180 kN at an eccentricity of 10 mm in a plane bisecting the thickness. Find the maximum and minimum intensities of stress in the section.

Solution. Given: Width (b) = 150 mm; Thickness (d) = 120 mm; Load (P) = 180 kN = 180×10^3 N and eccentricity (e) = 10 mm.



Maximum intensity of stress in the section

We know that area of the strut,

$$A = b \times d = 150 \times 120 = 18\,000\ mm^2$$

and maximum intensity of stress in the section,

$$\sigma_{max} = \frac{P}{A}\left(1 + \frac{6e}{b}\right) = \frac{180 \times 10^3}{18000}\left(1 + \frac{6 \times 10}{150}\right)$$

$$= 10\,(1 + 0.4) = 14\,N/mm^2 = \mathbf{14\,MPa}$$

Minimum intensity of stress in the section

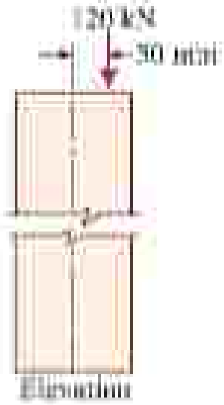
We also know that minimum intensity of stress in the section,

$$\sigma_{min} = \frac{P}{A}\left(1 - \frac{6e}{b}\right) = \frac{180 \times 10^3}{18000}\left(1 - \frac{6 \times 10}{150}\right)$$

$$= 10\,(1 - 0.4) = 6\,N/mm^2 = \mathbf{6\,MPa}$$

Example – 2: A rectangular column 200 mm wide and 150 mm thick is carrying a vertical load of 120 kN at an eccentricity of 50 mm in a plane bisecting the thickness. Determine the maximum and minimum intensities of stress in the section.

Solution. Given: Width (b) = 200 mm; Thickness (d) = 150 mm; Load (P) = 120 kN = 120×10^3 N and eccentricity (e) = 50 mm.



Maximum intensity of stress in the section

We know that area of the column,

$$A = b \times d = 200 \times 150 = 30\,000\ mm^2$$

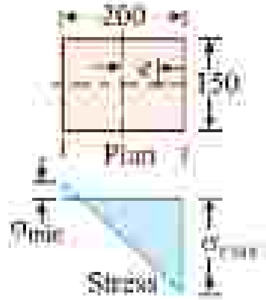
and maximum intensity of stress in the section,

$$\sigma_{max} = \frac{P}{A}\left(1 + \frac{6e}{b}\right) = \frac{120 \times 10^3}{30000}\left(1 + \frac{6 \times 50}{200}\right)$$

$$= 4\,(1 + 1.5) = 10\,N/mm^2 = \mathbf{10\,MPa}$$

Minimum intensity of stress in the section

We also know that minimum intensity of stress in the section,



$$\sigma_{min} = \frac{P}{A}\left(1 - \frac{6e}{b}\right) = \frac{120 \times 10^3}{30000}\left(1 - \frac{6 \times 50}{200}\right)$$

$$= 4\,(1 - 1.5) = -2\ N/mm^2 = 2\ \boldsymbol{MPa\ (tension)}$$

6.4 Buckling load computation using Euler's formula (no derivation) in Columns with various end conditions:

Buckling load:

- The maximum limiting load at which the column tends to have lateral displacement or tends to buckle is called buckling or crippling load. The buckling takes place about the axis having least moment of inertia.

Euler's formula:

- Euler's formula is used for calculating the critical load or buckling load or crippling load for a column or strut and is as follows.

$$P_{Euler} = \frac{\pi^2 EI}{L_e^{\,2}}$$

Where, P = Critical load

E = Young's modulus

I = Least M.I. of section of the column

L_e = Equivalent length of the column

Equivalent length of a column (L_e):

- The distance between adjacent points of inflection is called equivalent length or effective length or simple column length.

Types of End Conditions of Columns:

In actual practice, there are a number of end conditions, for columns. But we shall study the Euler's column theory on the following four types of end conditions.

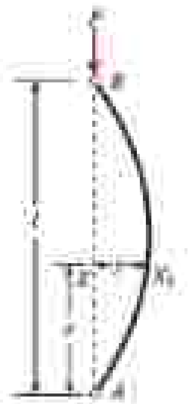
- Both ends hinged,
- Both ends fixed,
- One end is fixed and the other hinged, and
- One end is fixed and the other free

Columns with Both Ends Hinged or Pinned:

Equivalent length = Actual length

$$\Rightarrow L_e = l$$

$$P = \frac{\pi^2 EI}{L_e^{\,2}} = \frac{\pi^2 EI}{l^2}$$



Columns with One End Fixed and the Other Free:

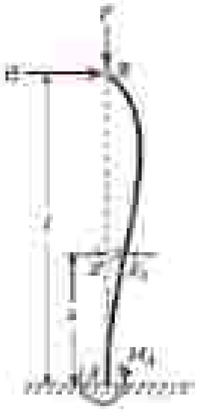
$$Here, L_e = 2l$$

$$P = \frac{\pi^2 EI}{L_e^{\,2}} = \frac{\pi^2 EI}{4l^2}$$

Columns with One End Fixed and the Other Hinged:

$$Here, L_e = \frac{l}{\sqrt{2}}$$

$$P = \frac{\pi^2 EI}{L_e^2} = \frac{2\pi^2 EI}{l^2}$$



Columns with Both Ends Fixed:

$$Here, L_e = \frac{l}{2}$$

$$P = \frac{\pi^2 EI}{L_e^2} = \frac{4\pi^2 EI}{l^2}$$

POSSIBLE SHORT TYPE QUESTIONS WITH ANSWER

01. What do you mean by column? (W – 2020)

- A column is a long vertical slender bar or vertical member, subjected to an axial compressive load and fixed rigidly at both ends.

02. Define eccentric load.

- A load, whose line of action does not coincide with the axis of a column or a strut, is known as an eccentric load.

03. Define buckling load.

- The maximum limiting load at which the column tends to have lateral displacement or tends to buckle is called buckling or crippling load. The buckling takes place about the axis having least moment of inertia.

04. Define equivalent length of a column.

- The distance between adjacent points of inflection is called equivalent length or effective length or simple column length.

POSSIBLE LONG TYPE QUESTIONS

01. What is meant by eccentric loading? Explain its effect on a short column. **(W – 2019)**
Hints: Refer article 6.1

02. Define buckling load. State formula for buckling load in column with various end condition. **(W – 2020)**
Hints: Refer article 6.4

CHAPTER NO. – 07

TORSION

LEARNING OBJECTIVES:

7.1 Assumption of pure torsion
7.2 The torsion equation for solid and hollow circular shaft
7.3 Comparison between solid and hollow shaft subjected to pure torsion.

Shaft:

- Shafts are usually cylindrical in section, solid or hollow. They are made of mild steel, alloy steel and copper alloys.
- It is a structural member used to transmit mechanical power from one place to other. Shaft is subjected to torsion.
- Shafts may be subjected to the following loads:
    1. Torsional load
    2. Bending load
    3. Axial load
    4. Combination of above three loads.

Torsion:

- A shaft or circular section is said to be in pure torsion when it is subjected to equal and opposite end couples whose axes coincide with the axis of the shaft.

7.1 Assumption of pure torsion:

- The torsion equation is based on the following assumptions:
- The material of the shaft is homogeneous, isotropic and perfectly elastic.
- The material obeys Hooke's law and the stress remains within limit of proportionality.
- The shaft circular in section remains circular after loading.
- A plane section of shaft normal to its axis before loading remains plane after the torques have been applied.
- The twist along the length of shaft is uniform throughout.
- The distance between any two normal cross section remains the same after the application of torque.
- Maximum shear stress induced in the shaft due to application of torque does not exceed its elastic limit value.

7.2 The torsion equation for solid and hollow circular shaft:

Torsion equation for solid circular shaft:

Let, T = Maximum twisting torque

D = Diameter of the shaft

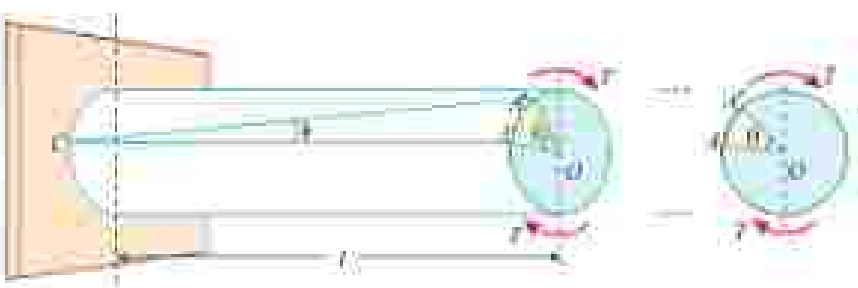
J_P = Polar moment of inertia

τ = Shear stress

G = Modulus of rigidity

θ = The angle of twist (radians)

l = Length of the shaft



- Consider a solid circular shaft fixed at one end and torque being applied at the other end. If a line CA is drawn on the shaft, it will be distorted to CA' on the application of the torque; thus, cross section will be twisted through angle θ and surface by angle ϕ.

- *Here, shear strain,*

$$\phi = \frac{AA'}{l}$$

- *Also,*

$$\phi = \frac{\tau}{G}$$

$$\therefore \frac{AA'}{l} = \frac{\tau}{G}$$

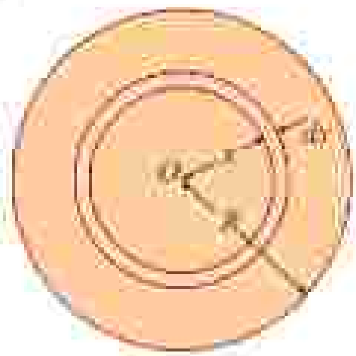
$$\Rightarrow \frac{R\theta}{l} = \frac{\tau}{G}$$

$$\left[\begin{matrix}\because AA' = R \times \theta \\ R \text{ being radius of the shaft}\end{matrix}\right]$$

$$\therefore \frac{\tau}{R} = \frac{G\theta}{l} \ldots\ldots\ldots (i)$$

- Consider an elementary ring of thickness dx at a radius x and let the shear stress at this radius be τ_x.

$$i.e.\ \ \tau_x = \frac{G\theta}{l}.x$$



- The turning force on the elementary ring

$$= \tau_x . 2\pi x . dx$$

- Turning moment due to this turning force,

$$dT = \tau_x . 2\pi x . dx \times x$$

- To get total turning moment integrating both sides, we get

$$\int dT = \int_0^R \tau_x . 2\pi x . dx \times x$$

$$\Rightarrow \int dT = 2\pi \int_0^R \frac{G\theta}{l} . x . x^2 . dx = 2\pi . \frac{G\theta}{l} \int_0^R x^3 . dx$$

$$\Rightarrow T = 2\pi . \frac{G\theta}{l} . \left[\frac{x^4}{4}\right]_0^R = 2\pi . \frac{\tau}{R} . \frac{R^4}{4}$$

$$\Rightarrow \boldsymbol{T = \tau . \frac{\pi R^3}{2} = \tau . \frac{\pi}{16} . D^3}$$

$$\ldots\ldots\ldots (Strength\ of\ solid\ shaft)$$

$$Or, \quad T = \frac{\tau}{R} . \frac{\pi R^4}{2} = \frac{\tau}{R} . I_P$$

$$\left[\because I_P = \frac{\pi}{32} . D^4 = \frac{\pi}{2} . R^4\right]$$

$$\therefore \frac{T}{I_P} = \frac{\tau}{R} \ldots\ldots\ldots (ii)$$

- From equation (*i*) and (*ii*), we have

$$\frac{T}{I_P} = \frac{\tau}{R} = \frac{G\theta}{l}$$

- This is called **torsion equation**.
- **Note:** From the relation,

$$\frac{T}{I_P} = \frac{\tau}{R}$$

$$\Rightarrow T = \tau \times \frac{I_P}{R}$$

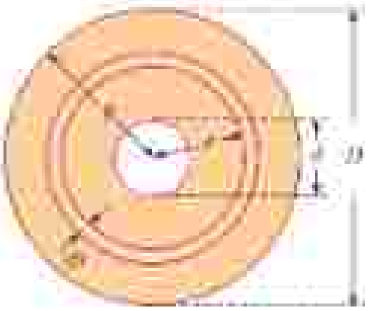
For a given shaft I_P and R are constants and I_P/R is thus a constant and is known as polar modulus of the shaft section.

$$Thus, \quad T = \tau \times Z_P$$

Torsion equation for hollow circular shaft:

- Torsion equation equally holds good for hollow circular shafts and can be established in the same way.
- Consider a hollow circular shaft subjected to a torque T
- Let, R = Outer radius of the shaft
  r = Inner radius of the shaft
- Consider an elementary ring of thickness *dx* at a radius x and let the shear stress at this radius be τ_x.

$$i.e. \quad \tau_x = \frac{G\theta}{l} \cdot x$$



- The turning force on the elementary ring

$$= \tau_x \cdot 2\pi x \cdot dx$$

- Turning moment due to this turning force,

$$dT = \tau_x \cdot 2\pi x \cdot dx \times x$$

- To get total turning moment integrating both sides, we get

$$\int dT = \int_r^R \tau_x \cdot 2\pi x \cdot dx \times x$$

$$\Rightarrow \int dT = 2\pi \int_r^R \frac{G\theta}{l} \cdot x \cdot x^2 \cdot dx = 2\pi \cdot \frac{G\theta}{l} \int_r^R x^3 \cdot dx$$

$$\Rightarrow T = 2\pi \cdot \frac{G\theta}{l} \cdot \left[\frac{x^4}{4}\right]_r^R = 2\pi \cdot \frac{\tau}{R} \cdot \left(\frac{R^4 - r^4}{4}\right)$$

$$\Rightarrow T = \frac{\tau}{R} \cdot \frac{\pi}{2} (R^4 - r^4) = \frac{\pi}{16} \cdot \tau \left(\frac{D^4 - d^4}{D}\right)$$

$$------(Strength\ of\ hollow\ shaft)$$

$$Or, \quad T = \frac{\tau}{R} \cdot I_P$$

$$\left[\because I_P = \frac{\pi}{32}(D^4 - d^4) = \frac{\pi}{2}(R^4 - r^4)\right]$$

$$\therefore \frac{T}{I_P} = \frac{\tau}{R} \text{.........}(iii)$$

- From equation (*i*) and (*iii*), we have

$$\frac{T}{I_P} = \frac{\tau}{R} = \frac{G\theta}{l}$$

Torsional rigidity:

- From the relation,

$$\frac{T}{I_P} = \frac{G\theta}{l}$$

- We have,

$$\theta = \frac{Tl}{GI_P}$$

- Since G, *l* and I_P are constant for a given shaft, θ the angle of twist is directly proportional to the twisting moment. The quantity $\frac{GI_P}{l}$ is known as torsional rigidity and is represented by k or μ.

- From the above relation, we have

$$k = \frac{GI_P}{l} = \frac{T}{\theta}$$

Power transmitted by the shaft:

- Consider a force *F* newton acting tangentially on the shaft of radius *R*. If the shaft due to this turning moment ($F \times R$) starts rotating at *N* r.p.m. then,

$$Work\ supplied\ to\ the\ shaft/sec = Force \times distance\ moved/sec$$
$$= F \times \frac{2\pi RN}{60}\ Nm/s$$

$$P = \frac{F \times 2\pi RN}{60} = \frac{2\pi NT}{60}\ watt$$

$$\text{.........} (\because T = F \times R)$$

Where, *T* = Mean/average torque

Example – 1: A circular shaft of 60 mm diameter is running at 150 r.p.m. If the shear stress is not to exceed 50 MPa, find the power which can be transmitted by the shaft.

Solution. Given: Diameter of the shaft (D) = 60 mm; Speed of the shaft (N) = 150 r.p.m. and maximum shear stress (τ) = 50 MPa = 50 N/mm².

- We know that torque transmitted by the shaft,

$$T = \frac{\pi}{16} \times \tau \times D^3$$

$$= \frac{\pi}{16} \times 50 \times 60^3$$

$$= 2.12 \times 10^6\ N.mm = 2.12\ kN.m$$

- and power which can be transmitted by the shaft,

$$P = \frac{2\pi NT}{60} = \frac{2\pi \times 150 \times 2.12}{60} = 33.3\ kW$$

Example - 2: A hollow shaft of external and internal diameters as 100 mm and 40 mm is transmitting power at 120 r.p.m. Find the power the shaft can transmit, if the shearing stress is not to exceed 50 MPa.

Solution. Given: External diameter (D) = 100 mm; Internal diameter (d) = 40 mm; Speed of the shaft (N) = 120 r.p.m. and allowable shear stress (τ) = 50 MPa = 50 N/mm².

- We know that torque transmitted by the shaft,

$$T = \frac{\pi}{16} \times \tau \times \left(\frac{D^4 - d^4}{D}\right)$$

$$= \frac{\pi}{16} \times 50 \times \left(\frac{100^4 - 40^4}{100}\right)$$

$$= 9.56 \times 10^6\ N.mm = 9.56\ kN.m$$

- and power which can be transmitted by the shaft,

$$P = \frac{2\pi NT}{60} = \frac{2\pi \times 120 \times 9.56}{60} = \mathbf{120\ kW}$$

Example – 3: A solid circular shaft of 100 mm diameter is transmitting 120 kW at 150 r.p.m. Find the intensity of shear stress in the shaft.

Solution. Given: Diameter of the shaft (D) = 100 mm; Power transmitted (P) = 120 kW and speed of the shaft (N) = 150 r.p.m.

- We know that power transmitted by the shaft (P),

$$120 = \frac{2\pi NT}{60} = \frac{2\pi \times 150 \times T}{60}$$

$$\Rightarrow T = \frac{120 \times 60}{2\pi \times 150} = 7.64\ kN.m$$

$$= 7.64 \times 10^6\ N.mm$$

- We also know that torque transmitted by the shaft (T),

$$7.64 \times 10^6 = \frac{\pi}{16} \times \tau \times D^3 = \frac{\pi}{16} \times \tau \times 100^3$$

$$\Rightarrow \tau = \frac{7.64 \times 16}{\pi} = 39\ N/mm^2 = \mathbf{39\ MPa}$$

Example – 4: A hollow shaft is to transmit 200 kW at 80 r.p.m. If the shear stress is not to exceed 60 MPa and internal diameter is 0.6 of the external diameter, find the diameters of the shaft.

Solution. Given: Power (P) = 200 kW; Speed of shaft (N) = 80 r.p.m.; Maximum shear stress (τ) = 60 MPa = 60 N/mm² and internal diameter of the shaft (d) = 0.6D (where D is the external diameter in mm).

- We know that torque transmitted by the shaft,

$$T = \frac{\pi}{16} \times \tau \times \left(\frac{D^4 - d^4}{D}\right) = \frac{\pi}{16} \times 60 \times \left(\frac{D^4 - (0.6D)^4}{D}\right)$$

$$= 10.3\ D^3\ N.mm = 10.3 \times 10^{-6}\ D^3\ kN.m$$

- We also know that power transmitted by the shaft (P),

$$200 = \frac{2\pi NT}{60} = \frac{2\pi \times 80 \times (10.3 \times 10^{-6}\ D^3)}{60} = 86.3 \times 10^{-6}\ D^3$$

$$\Rightarrow D^3 = \frac{200}{86.3 \times 10^{-6}} = 2.32 \times 10^6\ mm^3$$

$$\Rightarrow D = 1.32 \times 10^2 = \mathbf{132\ mm}$$

$$And, \qquad d = 0.6\,D = 0.6 \times 132 = \mathbf{79.2\ mm}$$

7.3 Comparison between solid and hollow shaft subjected to pure torsion:

(A) Comparison by strength:

- In this case it is assumed that both the shafts have same length, material, same weight and hence the same maximum shear stress.

- We know that,

$$T_S = \tau \times \frac{\pi}{16} \times (D_S)^3$$

$$T_H = \tau \times \frac{\pi}{16} \times \left[\frac{(D_H)^4 - (d_H)^4}{D_H}\right]$$

$$\therefore \frac{Strength\ of\ hollow\ shaft}{Strength\ of\ solid\ shaft} = \frac{T_H}{T_S} = \frac{\tau \times \frac{\pi}{16} \times \left[\frac{(D_H)^4 - (d_H)^4}{D_H}\right]}{\tau \times \frac{\pi}{16} \times (D_S)^3}$$

$$\Rightarrow \frac{T_H}{T_S} = \frac{(D_H)^4 - (d_H)^4}{D_H \cdot (D_S)^3} \ldots\ldots\ldots (i)$$

$$Let, \quad \frac{D_H}{d_H} = n$$

$$\Rightarrow D_H = n d_H$$

- Substituting $D_H = n d_H$ in equation (i), we get

$$\frac{T_H}{T_S} = \frac{n^4 d_H{}^4 - d_H{}^4}{n d_H D_S{}^3} = \frac{d_H{}^4(n^4 - 1)}{n d_H D_S{}^3} = \frac{d_H{}^3(n^4 - 1)}{n D_S{}^3} \ldots\ldots (ii)$$

- As the weight, material and length of both the shafts are same,

  Cross sectional area of solid shaft = Cross sectional area of hollow shaft

$$\frac{\pi}{4} D_S{}^2 = \frac{\pi}{4}\left(D_H{}^2 - d_H{}^2\right) \quad or \quad D_S = \sqrt{D_H{}^2 - d_H{}^2}$$

$$\Rightarrow D_S{}^3 = \left(D_H{}^2 - d_H{}^2\right)\sqrt{D_H{}^2 - d_H{}^2}$$

$$\Rightarrow D_S{}^3 = \left(n^2 d_H{}^2 - d_H{}^2\right)\sqrt{n^2 d_H{}^2 - d_H{}^2}$$

$$\Rightarrow D_S{}^3 = d_H{}^3(n^2 - 1)\sqrt{n^2 - 1}$$

- Substituting the value of $D_S{}^3$ in equation (ii), we get

$$\frac{T_H}{T_S} = \frac{d_H{}^3(n^4 - 1)}{n\, d_H{}^3(n^2 - 1)\sqrt{n^2 - 1}}$$

$$= \frac{(n^2 + 1)(n^2 - 1)}{n\,(n^2 - 1)\sqrt{n^2 - 1}} = \frac{n^2 + 1}{n\sqrt{n^2 - 1}}$$

- Since $D_H > d_H$ and $\frac{D_H}{d_H} = n$, it is thus obvious that the value of 'n' is greater than unity.
- Suppose, $n = 2$

  Then,

$$\frac{T_H}{T_S} = \frac{n^2+1}{n\sqrt{n^2-1}} = \frac{2^2+1}{2\sqrt{2^2-1}} = 1.44$$

- This shows that the torque transmitted by the hollow shaft is greater than the solid shaft, thereby proving that the hollow shaft is stronger than the solid shaft.

(B) **Comparison by weight:**

- In this case it is assumed that both the shafts have same length and material. Now, if the torque applied to both shafts is same, then, the maximum shear stress will also be same in both the cases.
- Now,

$$\frac{Weight\ of\ hollow\ shaft}{Weight\ of\ solid\ shaft} = \frac{W_H}{W_S} = \frac{A_H}{A_S}$$

$$= \frac{\frac{\pi}{4}\left(D_H^{\ 2} - d_H^{\ 2}\right)}{\frac{\pi}{4}D_S^{\ 2}} = \frac{D_H^{\ 2} - d_H^{\ 2}}{D_S^{\ 2}} \ldots\ldots\ldots (i)$$

$$Let, \quad \frac{D_H}{d_H} = n$$

$$\Rightarrow D_H = nd_H$$

- Substituting $D_H = nd_H$ in equation (i), we get

$$\frac{W_H}{W_S} = \frac{n^2 d_H^{\ 2} - d_H^{\ 2}}{D_S^{\ 2}} = \frac{d_H^{\ 2}(n^2-1)}{D_S^{\ 2}} \ldots\ldots\ldots (ii)$$

- Torque applied in both the cases is same i.e., $T_S = T_H$

$$\tau \times \frac{\pi}{16} \times (D_S)^3 = \tau \times \frac{\pi}{16} \times \left[\frac{(D_H)^4 - (d_H)^4}{D_H}\right]$$

$$\Rightarrow D_S^{\ 3} = \frac{(D_H)^4 - (d_H)^4}{D_H} = \frac{n^4 d_H^{\ 4} - d_H^{\ 4}}{nd_H} = \frac{d_H^{\ 3}(n^4-1)}{n}$$

$$\Rightarrow D_S = d_H\left[\frac{n^4-1}{n}\right]^{1/3} \Rightarrow D_S^{\ 2} = d_H^{\ 2}\left[\frac{n^4-1}{n}\right]^{2/3}$$

- Substituting the value of $D_S^{\ 2}$ in equation (ii), we have

$$\frac{W_H}{W_S} = \frac{d_H^{\ 2}(n^2-1)}{d_H^{\ 2}\left[\frac{n^4-1}{n}\right]^{2/3}} = \frac{(n^2-1)n^{2/3}}{(n^4-1)^{2/3}}$$

- If $n = 2$ then,

$$\frac{W_H}{W_S} = \frac{(2^2-1)\times(2)^{2/3}}{(2^4-1)^{2/3}} = 0.7829$$

- It shows that for same material, length and given torque, weight of hollow shaft will be less. So, hollow shafts are economical compared to solid shafts as regards torsion.

POSSIBLE SHORT TYPE QUESTION WITH ANSWERS:

01. Define torsion. (W – 2020)

- A shaft or circular section is said to be in pure torsion when it is subjected to equal and opposite end couples whose axes coincide with the axis of the shaft.

02. What is polar moment of inertia? (W – 2019)

- The moment of inertia of a plane area, with respect to an axis perpendicular to the plane of the figure, is called polar moment of inertia with respect to the point, where the axis intersects the plane.

03. Define shaft.(W-2021)

Ans: It is a structural member used to transmit mechanical power from one place to other. Shaft is subjected to torsion.

04. What is the function of shaft.(W-2022)

Ans: Its function is to transmit power from a driving device, such a motor or engine through a machine.

POSSIBLE LONG TYPE QUESTIONS:

01. A solid circular shaft of 80 mm diameter is required to transmit power at 120 r.p.m. If the shear stress is not to exceed 40 MPa, find the power transmitted by the shaft. **(W – 2019)**

Hints: Refer Example – 01

02. Find the maximum shear stress induced in a solid circular shaft of diameter 15 cm when the shaft transmits 150 kW power at 180 r.p.m. **(W – 2020)**

Hints: Refer Example – 03

03. What are the assumptions of pure torsion?(W-2022)

Hints: Refer Article – 7.1

04. Derive $\frac{T}{I_p} = \frac{\tau}{R} = \frac{G\theta}{l}$ for pure torsion.

Where, T = Maximum twisting torque

R = Radius of the shaft

I_P = Polar moment of inertia

τ = Shear stress

G = Modulus of rigidity

θ = The angle of twist (radians)

l = Length of the shaft

Hints: Refer Article – 7.2

05. A circular bar is subjected an axial pull of 120KN. If the maximum intensity of shear stress on any oblique plane is not to exceed 55MN/M^2, find the diameter.(W-2021)

6. A solid circular shaft of 100MM diameter is transmitting 120KW at 150rpm find the intensity of shear stress in the shaft.(W-2022)