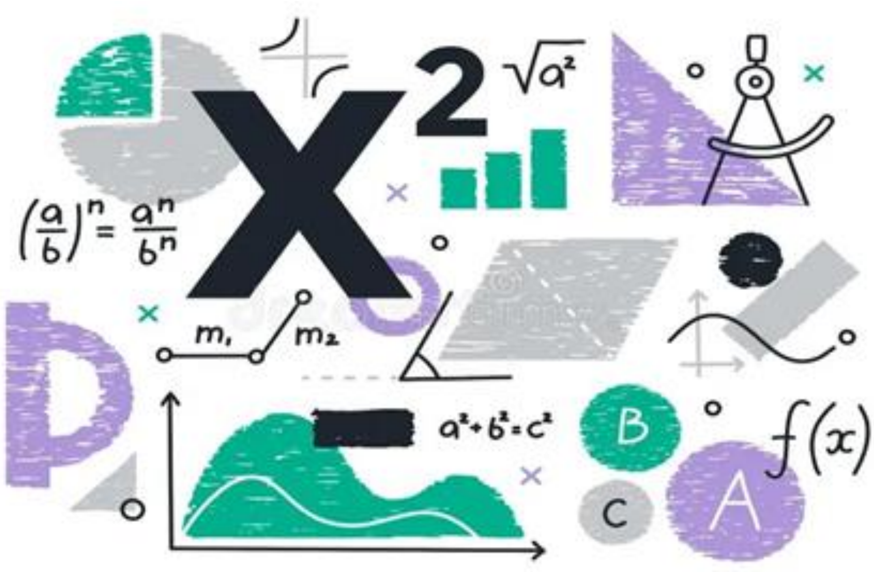
BHADRAK ENGINEERING SCHOOL & TECHNOLOGY (BEST),
ASURALI, BHADRAK

Mathematics-II

(Th- 03)

(As per the 2024-25 syllabus prepared by the SCTE&VT, Bhubaneswar, Odisha)



SECOND Semester

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MATHEMATICS-II

CHAPTER-WISE DISTRIBUTION OF PERIODS & MARKS

Sl No	Unit No	Topics	Periods Actually Needed	Expected Mark
01	I	Determinants and Matrices	10	15
02	II	Integral Calculus	20	25
03	III	Co-Ordinate Geometry	15	25
04	IV	Vector Algebra	10	15
05	V	Differential Equations	05	10
	Total		60	90

UNIT-I

DETERMINANTS AND MATRICES

LEARNING OBJECTIVES :

- *Elementary Properties of determinants up to 3rd order*
- *Consistency of equations*
- *Cramer's Rule*
- *Algebra of matrices*
- *Inverse of a matrix*
- *Matrix inverse method to solve a system of linear equations in 3 variables*

DETERMINANT :

Introduction: A determinant is defined as a function from set of square matrices to the set of real numbers.

. Every square matrix is associated with a number, called its determinant,

denoted by det (A) or |A| or Δ.

. Only square matrices have determinants.

If the linear equations

ax + b = 0

cx + d =0

have the same solution then $\frac{b}{a} = \frac{d}{c}$

or (ad – bc) = 0

The expression (ad – bc) is called a determinant and is denoted by the symbol $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ where a, b, c , d are called elements of the determinant . The elements in the horizontal direction form rows and elements in the vertical direction form columns.

The det. Of A is written as |A| and is read as det. of A not modulus of A .

The above determinant has two rows and two columns, so it is called a determinant of 2nd order.

Similarly, $\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}$ Is a 3rd order determinant.

Expansion of a determinant:

(i) The sum of the product of element of any row (column) with their corresponding cofactors is always equal to the value of the determinant.

(ii) The sum of the product of elements of any row (column) with the cofactors of the other row (column) is always equal to zero.

Minors :-

The minor of an element a_{ij} is the det. Obtained by omitting the i^{th} row and j^{th} column of a det. in which a particular element occurs is called the minor of that element. Generally minor of an element is denoted by M_{ij} or A_{ij} .

Minor of an element in a 3rd Order determinant is a 2nd order determinant. Therefore, in the above 3rd determinant Minor of 'a' is $\begin{vmatrix} q & r \\ y & z \end{vmatrix}$.

The Minor of b is $\begin{vmatrix} p & r \\ x & z \end{vmatrix}$ and c is $\begin{vmatrix} p & q \\ x & y \end{vmatrix}$. Similarly, we can find out the Minors for p, q , r and x , y ,z .

If A = $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$

And Δ stands for the value of the determinant then

Det (A) or |A| or Δ = ad -bc

And for 3rd order det. ,

If Δ = $\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}$ then expanding along 1st row , we get ,

= a $\begin{vmatrix} q & r \\ y & z \end{vmatrix}$ - b $\begin{vmatrix} p & r \\ x & z \end{vmatrix}$ + c $\begin{vmatrix} p & q \\ x & y \end{vmatrix}$

= a M_{11} – b M_{12} + c M_{13} , where M_{11}, M_{12} , M_{13} are minors of a, b , c respectively .

*In the expansion of the determinant of 3rd order the signs with which the elements are multiplied may be remembered by the following formula.

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

Example: Find the value of $\begin{vmatrix} 5 & 2 & 1 \\ 3 & 0 & 2 \\ 8 & 1 & 3 \end{vmatrix}$.

Solution: The value of the given determinant = 5 $\begin{vmatrix} 0 & 2 \\ 1 & 3 \end{vmatrix} - 2 \begin{vmatrix} 3 & 2 \\ 8 & 3 \end{vmatrix} + 1 \begin{vmatrix} 3 & 0 \\ 8 & 1 \end{vmatrix}$

= 5 (0 - 2) -2 (9- 16) + 1 (3- 0) = -10 + 14 + 3 = 7 .

 # The determinant can be expanded by taking the elements of any row or any column.

Cofactors :-

The cofactor of an element in a determinant is its coefficient in the expansion of the determinant.

It is therefore equal to the corresponding Minor with proper sign. Cofactors are generally denoted by C_{ij} .

$C_{ij} = (-1)^{i+j} M_{ij}$

Where C_{ij} and M_{ij} are respectively co-factor and minor of element a_{ij} .

Thus in the determinant, $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

Cofactor of a_{11}=$(-1)^{1+1}$ $\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$, Cofactor of a_{12} = $(-1)^{1+2} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$, Cofactor of a_{13} = $(-1)^{1+3} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

Similarly we can find out the cofactors of other elements .

ELEMENTARY PROPERTIES OF DETERMINANTS UPTO 3rd ORDER:-

1. The Value of a determinant does not change if the rows and columns of a determinant are interchanged

2. If two adjacent rows or columns of a determinant are interchanged then the value of the determinant is changed by sign but the absolute value remains same.

3. If two rows or columns of a determinant are identical then the value of the determinant is zero .

4. If each element of any row or any column of a determinant is multiplied by same factor then the determinant is multiplied by that factor .

5. If every element of any row or column of a determinant can be expressed as sum of two number then the determinant can be expressed as the sum of two determinants.

6. A determinant remains unchanged by adding 'K' times the element of any row or column to corresponding element of any other row or column where 'K' is any number.

EXAMPLES :

1. Evaluate $\begin{vmatrix} sin\alpha & -cos\alpha \\ cos\alpha & sin\alpha \end{vmatrix}$

Solution : $sin^2\alpha + cos^2\alpha = 1$

2. Find the value of $\begin{vmatrix} 4 & -1 \\ 3 & 2 \end{vmatrix}$.

Solution : (4) (2) - (3) (-1) = 8+3 = 11

3. Without expanding prove that $\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$ = (a – b) (b- c) (c- a)

Solution : Applying $R_2 \rightarrow R_2 - R_1$, $R_3 \rightarrow R_3 - R_1$, we get

$$\Delta = \begin{vmatrix} 1 & a & bc \\ 0 & b-a & c(a-b) \\ 0 & c-a & b(a-c) \end{vmatrix}$$

Taking factors (b -a) & (c- a) common from R_2 & R_3 , respectively , we get

$$\Delta = (b -a)(c- a) \begin{vmatrix} 1 & a & bc \\ 0 & 1 & -c \\ 0 & 1 & -b \end{vmatrix}$$

$= (b - a)(c- a)[(-b + c)]$ (expanding along 1st column)

$= (a - b) (b – c) (c- a)$ (proved)

4. Prove that $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} = 0$

Solution ; L. .S = $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$

$= \begin{vmatrix} a-b+b-c+c-a & b-c & c-a \\ b-c+c-a+a-b & c-a & a-b \\ c-a+a-b+b-c & a-b & b-c \end{vmatrix}$ Applying $c_1 \rightarrow c_1 + c_2 + c_3$

$= \begin{vmatrix} 0 & b-c & c-a \\ 0 & c-a & a-b \\ 0 & a-b & b-c \end{vmatrix} = 0$ (proved)

5 . prove that $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a + b + c)^3$

Solution : L.H.S = $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$

$= \begin{vmatrix} a-b-c+2b+2c & 2a+b-c-a+2c & 2a+2b+c-a-b \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$ **,(** replacing R_1 by $R_1 + R_2 + R_3$)

$= = \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c) \begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$

$= (a+b+c) \begin{vmatrix} 0 & 0 & 1 \\ b+c+a & -(a+b+c) & 2b \\ 0 & a+b+c & c-a-b \end{vmatrix}$, (Replacing C_1 by $C_1 - C_2$ and C_2 by $C_2 - C_3$)

$= (a+b+c) \begin{vmatrix} (a+b+c) & -(a+b+c) \\ 0 & (a+b+c) \end{vmatrix}$

$= (a+b+c)(a+b+c)^2 = (a+b+c)^3$ = R.H.S (Proved)

CONSISTENCY OF EQUATIONS:

Let us consider a system of linear equations be

$$a_1x + b_1y + c_1z = d_1\Delta$$

$$a_2x + b_2y + c_2z = d_2$$

$$a_3x + b_3y + c_3z = d_3$$

Then x= $\frac{\Delta_x}{\Delta}$, y = $\frac{\Delta_y}{\Delta}$ and z = $\frac{\Delta_z}{\Delta}$, where Δ_x, Δ_y *and* Δ_z *have their usual meaning.*

Note :

(i) If $\Delta \neq 0$ and atleast one of Δ_x, Δ_y *and* $\Delta_z \neq 0$, then given system of equations is consistent and has unique non-trivial solution.

(ii) If $\Delta \neq 0$ and $\Delta_x = \Delta_y = \Delta_z = 0$ then the then given system of equations is consistent and has trivial solution.

(iii) If $\Delta = \Delta_x = \Delta_y = \Delta_z = 0$, then the then given system of equations is consistent and has infinite solutions.

(iv) If $\Delta = 0$ but atleast one of Δ_x, Δ_y , Δ_z is not zero then equations are inconsistent and have no solution.

CRAMER'S RULE:

Cramer's Rule is an explicit formula for the solution of a system of linear equations with as many equations as unknown. It is applicable only if the value of the determinant is non-zero. Here, we restrict our study to the equations of two variables .

The solution of two equations

a_1 X + b_1 y = d_1

a_2 x + b_2 y = d_2

are given by X = $\frac{\Delta_x}{\Delta}$, y = $\frac{\Delta_y}{\Delta}$ where Δ = $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$, Δ_x = $\begin{vmatrix} d_1 & b_1 \\ d_2 & b_2 \end{vmatrix}$ and Δ_y = $\begin{vmatrix} a_1 & d_1 \\ a_2 & d_2 \end{vmatrix}$.

Ēxample : Solve the following by Cramer's Rule

2x - y = 2

3x + y = 13

Here Δ = $\begin{vmatrix} 2 & -1 \\ 3 & 1 \end{vmatrix} = 2 + 3 = 5$, $\Delta_x = \begin{vmatrix} 2 & -1 \\ 13 & 1 \end{vmatrix}$ = 2+13 = 15 and $\Delta_y = \begin{vmatrix} 2 & 2 \\ 3 & 13 \end{vmatrix}$ = 26 – 6 = 20

∴ x = $\frac{\Delta_x}{\Delta}$ = $\frac{15}{5}$ = 3 , y = $\frac{\Delta_y}{\Delta}$ = $\frac{20}{5}$ = 4

MATRIX :- A matrix is a rectangular array of numbers arranged in rows and columns. If there are 'm' rows and 'n' columns in a matrix then it is called an 'm' by 'n' Matrix or a matrix of order m× n. The first letter in m

× n denotes the number of rows and the second letter n denotes the number of columns. Generally, the capital letters of English alphabet are used to denote matrices and the actual matrix is enclosed in parentheses.

Hence $A = \begin{bmatrix} a_{11} & a_{12} & --- & a_{1n} \\ a_{21} & a_{22} & --- & a_{2n} \\ -- & --- & --- & -- \\ a_{m1} & a_{m2} & --- & a_{mn} \end{bmatrix}$

is a matrix of order (m × n) and a_{ij} denotes the element in the i^{th} row and j^{th}column . For example a_{23} is the element in the 2nd row and 3rd column . Thus the matrix A may be written as (a_{ij}) where i takes values from 1 to m to represent row and j takes values from 1 to n to represent column.

if m=n then the matrix A is called a square matrix of order n by n. Thus

$A = \begin{bmatrix} a_{11} & a_{12} & --- & a_{1n} \\ a_{21} & a_{22} & --- & a_{2n} \\ -- & -- & --- & -- \\ a_{n1} & a_{n2} & --- & a_{nn} \end{bmatrix}$ is a square matrix of order n.

TYPES OF MATRICES:

1. **Row Matrix**: A matrix with a single row is called a row matrix. For Ex. $[1 \quad 2]$, $(a \quad b \quad c)$ are matrices of order (1×2) *and* (1×3)

2. **Column Matrix** : A matrix with a single column is called a column matrix. Ex. :- the matrices $\begin{bmatrix} a \\ b \end{bmatrix}$, $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ are column matrices of order (2×1) *and* (3×1) respectively.

3. **Square Matrix**: A matrix in which number of rows is equal to number of columns is called a square matrix.

4. **Diagonal Matrix**: A square Matrix in which the non-diagonal elements are all zero is called a diagonal matrix

5. **Scalar Matrix**: A diagonal Matrix in which the diagonal elements are all equal is called a scalar matrix.

6. **Unit Matrix** : The square Matrix whose elements on its main diagonal (left top to right bottom) are all unity is called a unit matrix. It is denoted by I and it may be of any order.

$Thus\ (1),\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are unit matrices of order 1,2 ,3 respectively .

7. **Zero Matrix**: A matrix in which all the elements are all zero is called a zero matrix or null matrix denoted by (o). The zero matrix may be of any order . Thus $[0]$, $[0 \quad 0]$, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ are all zero matrices .

8. **Singular Matrix**: A square matrix whose determinant value is zero is called a singular matrix .

For example, $\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$ is a singular matrix. $\begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix} = 0$

9. **Non-singular Matrix**: A square matrix whose determinant value is not zero is called a non- singular matrix.

For example, $\begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$ is a non-singular matrix. As $\begin{vmatrix} 2 & 1 \\ 4 & 3 \end{vmatrix}$ = 6 -4 = 2 ≠ 0

10**. Transpose of Matrix :** The transpose of a matrix A is the matrix obtained from A by changing its rows into columns and columns into rows. It is denoted by A' or A^T

ALGEBRA OF MATRICES:

Equality of two matrices:

Two matrices A and B are said to be equal if and only if

1. Order of A and B are same .

2. Each element of A is equal to the corresponding element of B .

Addition of matrices:

The sum of two matrices A &B is the matrix such that each of its elements is equal to the sum of the corresponding elements of A and B . The sum is denoted by A + B. Thus the addition of matrices is defined if they are of same order and is not defined when they are of different orders. A, B & A+B are of same order.

Subtraction of Matrices: The subtraction of two matrices A & B of the same order is defined as A- B = A+(-B)

Product of a matrix and a scalar: The product of a scalar m and a matrix A is denoted by a mA, is the matrix each of whose elements is m times the corresponding element of A.

Example. If A = $\begin{pmatrix} 2 & 1 & 3 \\ -1 & 0 & 4 \\ 3 & -2 & 1 \end{pmatrix}$ then 3A = $\begin{pmatrix} 3\times 2 & 3\times 1 & 3\times 3 \\ 3\times -1 & 3\times 0 & 3\times 4 \\ 3\times 3 & 3\times -2 & 3\times 1 \end{pmatrix}$ = $\begin{pmatrix} 6 & 3 & 9 \\ -3 & 0 & 12 \\ 9 & -6 & 3 \end{pmatrix}$

Product of two matrices:- The product of two matrices A & B **(where the number of columns in A is equal to the number of rows in B)** is the matrix AB whose element in the i^{th} row and j^{th} column is the sum of the products formed by multiplying each element in the i^{th}row of A and the corresponding element in the j^{th} column of B.

Let A be an (m× k) matrix and B be a (k ×n) matrix. The product of A and B, denoted by AB is the (m×n) matrix with $(\text{i}, \text{j})^{th} entry$ equal to the sum of the products of a corsresponding elements from i^{th} row of A and j^{th} column of B.

For Example-1. if A = $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$,B = $\begin{pmatrix} x & y \\ u & v \end{pmatrix}$ then AB = $\begin{pmatrix} ax+bu & ay+bv \\ cx+du & cy+dv \end{pmatrix}$

Example - 2. if A = $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$, B = $(x \;\; y \;\; z)$ then AB = $\begin{pmatrix} ax & ay & az \\ bx & by & bz \\ cx & cy & cz \end{pmatrix}$

Example -3. if A = $(a \;\; b \;\; c)$, B = $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ then AB = $(ax + by + cz)$

Properties of Matrix Multiplication :

(I) The multiplication of matrices is not necessarily commutative i.e, if A and B are two matrices then AB is not equal to BA .

(II) The multiplication of matrices is associative i.e, if A,B,C are three matrices then (AB)C = A(BC), Provided the products are defined.

(III) The identity matrix for multiplication for the set of all square matrices of a given order is the unit matrix of the same order

(IV) Let A and B be 2 matrices such that the product AB is defined. Then A = 0 or B=0 or A=0= B always implies that AB equal to 0. Conversely AB=0 does not always imply that A =0 or B= 0 or A=0= B.

(V) The cancellation law does not hold for matrix multiplication, i.e CA=CB does not necessarily imply A equal to B.

(VI) The distributive laws hold good for matrices. If A, B & C are three matrices then A (B+C)= AB + AC , (A+B)C= AC + BC provided the addition and multiplication in above equations are defined .

System of algebraic equations can be expressed in the form of matrices .

. The values of variables satisfying all the linear equations in the system is called solution of the system of linear equations.

. If the system of linear equations has a unique solution, this unique solution is called **determinant** of solution.

Cofactor of $a_{11}=(-1)^{1+1}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$,

Cofactor of $a_{12} = (-1)^{1+2}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$,

Cofactor of $a_{13} = (-1)^{1+3}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

Similarly we can find out the cofactors of other elements .

<u>INVERSE OF A MATRIX:</u> If A is a non-singular matrix, then it's inverse matrix A^{-1} is defined as

$A^{-1} = \frac{adj\ A}{|A|}$, Where |A| is the determinant of matrix A and adj A is the adjoint of matrix A .

Transpose of a matrix : Transpose of a m$\times n$ matrix A is the matrix of order n$\times$ m obtained by interchanging the rows and columns of A . The transpose of the matrix A is written as A' or A^T.

For example if A = $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ then $A^T = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$

Adjoint of a matrix :

The adjoint of a matrix A is the transpose of the matrix obtained by replacing each element a_{ij} in A by its cofactor A_{ij} . The adjoint of A is written as adj A . Thus if

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \text{ then adj } A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}.$$

Example : Find inverse of the matrix $A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{pmatrix}$.

Solution : Expanding along R_1, $|A| = 0 - 1\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} + 2\begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}$ = - (1 -9)+ 2 (1-6) = 8 -10 = -2 $\neq$ 0

So, the inverse of A exists. Let C_{ij} and M_{ij} are cofactors and Minors of A.

Hence $C_{ij} = (-1)^{i+j} M_{ij}$

Cofactor of 0 , $C_{11} = (-1)^{(1+1)} \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix}$ = +(2-3) = -1

Cofactor of 1 , $C_{12} = (-1)^{(1+2)} \begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix}$ = - (1-9) = 8

Cofactor of 2 , $C_{13} = (-1)^{(1+3)} \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix}$ = +(1 - 6) = -5

Similarly cofactor of 1, $C_{21} = (-1)^{(2+1)} \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix}$ = -(1-2) = 1

Cofactor of 2 , $C_{22} = (-1)^{(2+2)} \begin{vmatrix} 0 & 2 \\ 3 & 1 \end{vmatrix}$ = +(0-6) = -6

cofactor of 3, $C_{23} = (-1)^{(2+3)} \begin{vmatrix} 0 & 1 \\ 3 & 1 \end{vmatrix}$ = -(0-3) = 3

cofactor of 3 , $C_{31} = (-1)^{(3+1)} \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix}$ = +(3-4)= -1

cofactor of 1 , $C_{32} = (-1)^{(3+2)} \begin{vmatrix} 0 & 2 \\ 1 & 3 \end{vmatrix}$ = -(0-2) = 2

cofactor of 1 , $C_{33} = (-1)^{(3+3)} \begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix}$ = +(0-1) = -1

we have ,

$$\text{adj } A = \begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{\text{adj } A}{|A|} = \frac{\begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix}}{-2} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 4 & -3 & 1 \\ -\frac{5}{2} & \frac{3}{2} & -\frac{1}{2} \end{bmatrix} \qquad \text{(ANS)}$$

MATRIX INVERSE METHOD TO SOLVE A SYSTEM OF LINEAR EQUATIONS IN 3 VARIABLES:

Suppose we have the following system of equations

$a_1x + b_1y + c_1z = d_1$

$a_2x + b_2y + c_2z = d_2$

$a_3x + b_2y + c_3z = d_3$

Where A = $\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$, x = $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and B = $\begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$

The equation in the matrix form becomes AX = B $\Longrightarrow$ X = A^{-1}B , if $A \neq 0$

$$\Longrightarrow X = \frac{adj\,A}{|A|} . B$$

Again for , $a_1x + b_1y = d_1$

$a_2x + b_2y = d_2$, Where A = $\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$, x = $\begin{pmatrix} x \\ y \end{pmatrix}$ and B = $\begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$; the system of equations can be written as AX = B ,

The solution being X = A^{-1}B = $\frac{adj\,A}{|A|}$. B

Example : Solve the following system of equations by Matrix method **:**

x– y + z = 4 , 2x + y – 3z = 0 , x + y + z = 2

Solution : Here A = $\begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}$, X = $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$, B = $\begin{pmatrix} 4 \\ 0 \\ 2 \end{pmatrix}$

|A| = 1(1 + 3) – (-1) (2 + 3) + 1 (2- 1) = 4 + 5 + 1 = 10 $\neq 0$

$c_{11} = +(1+3) = 4$, $c_{12} = -(2+3) = -5$ $c_{13} == (2-1) = 1$

c_{21}= - (-1 -1) = +2 c_{22} = + (1 -1) = 0 c_{23} = - (1 +1) = -2 ,

$c_{31} = +(3-1) = 2$, $c_{32} = -(-3-2) == 5$, $c_{33} = +(1+2) = 3$

Co-factor Matrix of A = $\begin{bmatrix} 4 & -5 & 1 \\ 2 & 0 & -2 \\ 2 & 5 & 3 \end{bmatrix}$

So, AdjA = $\begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}$

X = A^{-1}B => X = $\frac{AdjA}{|A|}$. B = $\frac{\begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}}{10} . \begin{pmatrix} 4 \\ 0 \\ 2 \end{pmatrix}$ = $\begin{bmatrix} \frac{2}{5}.4 + \frac{1}{5}.0 + \frac{1}{5}.2 \\ \frac{-1}{2}.4 + 0.0 + \frac{1}{2}.2 \\ 1\frac{11}{10}.4 - \frac{1}{5}.0 + \frac{3}{10}.2 \end{bmatrix}$ = $\begin{bmatrix} \frac{8}{5} + \frac{2}{5} \\ -2 + 1 \\ \frac{2}{5} + \frac{1}{5} \end{bmatrix}$ = $\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ = = $\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ => x = 2 , y = -1 & z = 1 (ans)

POSSIBLE SHORT TYPE QUESTIONS WITH ANSWER :

1. Find the cofactors of each element in = $\begin{vmatrix} 2 & 5 \\ 3 & -4 \end{vmatrix}$

Sol^n : Cofactor of 2 is +(-4) = - 4
Cofactor of 5 is –(3) = -3
Cofactor of 3 is - (5) = - 5
Cofactor of -4 is +(- 2) = - 2

2. Evaluate $\begin{vmatrix} 2 & 3 & 5 \\ 3 & -1 & 0 \\ 1 & 0 & 0 \end{vmatrix}$.

Sol^n : $\begin{vmatrix} 2 & 3 & 5 \\ 3 & -1 & 0 \\ 1 & 0 & 0 \end{vmatrix}$ = 2 (0 - 0) -3 (0 - 0) + 5 (0 + 1) = 0 – 0 + 5 = 5 (Ans)

3. Find the maximum value of $\begin{vmatrix} sin^2x & sinx.cosx \\ -cosx & sinx \end{vmatrix}$ (W-20)

Sol^n : $\begin{vmatrix} \sin^2 x & \sin x.\cos x \\ -\cos x & \sin x \end{vmatrix}$ = $\sin^3$x - (- sinx.cos^2x) = $\sin^3$x + sinx.cos^2x = sinx ($\sin^2$x + cos^2x)

= sin x .1 = sin x

As sin x has maximum value +1 hence the determinant has also maximum value +1. (Ans)

4. Find x & y if $\begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Solution: $\begin{bmatrix} 2x+y \\ 3x+4y \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ ⇒ 2x + y = 1 .……... (1)

and 3x + 4y = -1 ……… (2)

2× eq^n(2) ⇒ (6x + 8y) = -2 ..……... (3)

3 × eq^n(1) ⇒ (6x + 3y) = 3 ..……… (4)

Subtracting eq^n(4) from eq^n(3) , we get , 5y = (-2-3) = -5 => Y= -1

So, x = (1-3y)/2 = {1- 3(-1)}/2 = 4/2 = 2

∴ x = 2 and y = -1 .

5. If [3 4 2] B = [3 5 4 7 2] then find the order of B . (2010-w , 2014 w)

sol^n**:** [3 4 2] has order 1x3 and [3 5 4 7 2] has order 1x5, so B must be of order 3x5 .

6. Write down the matrix $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ if a_{ij} = 2i +3j (2008w)

$\boldsymbol{sol^n}$: for a_{11}, i=1 , j=1 so, a_{11} = 2(1) + 3(1) = 5

For a_{12} , i =1, j =2 so, a_{12} = 2(1)+ 3(2) = 2 + 6 = 8

For $a_{21}, i = 2 . j = 1$ so , a_{21} = 2(2)+ 3(1) =7

For $\boldsymbol{a_{22}}$, i = 2 , j = 2 so , $\boldsymbol{a_{22}}$ = 2(2)+ 3(2) = 4 + 6 = 10

So, the given matrix is $\begin{bmatrix} 5 & 8 \\ 7 & 10 \end{bmatrix}$.

7. Find the adjoint of the matrix $\begin{bmatrix} 1 & -1 \\ 3 & 4 \end{bmatrix}$.

$\boldsymbol{sol^n}$: Cofactor of 1 , C_{11} = = +4 , Cofactor of -1 , C_{12} = -(3) ,

cofactor of 3 , C_{21} = -(-1) = 1 ,cofactor of 4, C_{22} = 1

so, adjoint of given matrix is , $\begin{bmatrix} 4 & -3 \\ 1 & 1 \end{bmatrix}^T = \begin{bmatrix} 4 & 1 \\ -3 & 1 \end{bmatrix}$.

8. If A = $\begin{bmatrix} 2 & 4 \\ 3 & 13 \end{bmatrix}$ and I = $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ then find A- αI . [W-12]

$\boldsymbol{sol^n}$**:** αI = α $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ = $\begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix}$ A- αI = $\begin{bmatrix} 2 & 4 \\ 3 & 13 \end{bmatrix}$ - $\begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix}$ = = $\begin{bmatrix} 2-\alpha & 4 \\ 3 & 13-\alpha \end{bmatrix}$

POSSIBLE LONG TYPE QUESTIONS

1. Prove that $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-b-a \end{vmatrix} = (a+b+c)^3$ (w – 20)

2. Without expanding prove that $\begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix}$ = abc (a- b) (b – c)(c- a) . (w – 05)

3. Without expanding prove that $\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$ = (x - y) (y - z)(z – x) . (W- 15)

4. Solve the following by Cramer's Rule (W- 20)

   2x - 3y= 7

3x - 2y = 3

5. Find the adjoint of the matrix $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 1 & 3 & 1 \end{bmatrix}$ (W-12)

6. Find the inverse of the followings

(i) $\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$ (W-08) (ii) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 1 & 0 & 2 \end{bmatrix}$ (W-14)

(iii) $\begin{bmatrix} 3 & -2 & 3 \\ 2 & 1 & -1 \\ 4 & -3 & 2 \end{bmatrix}$ (W-10)

7. Solve the following equations by Matrix method (W-10)

x – y + z = 4 , 2x + y -3z = 0 , x + y + z = 2

UNIT-II

INTEGRAL CALCULUS

Learning Objectives:

- Integration as an Inverse operation of differentiation.
- Simple Integration by substitution, by parts and by partial fractions (for linear factors only).
- Use of formulas $\int_0^{\frac{\pi}{2}} \sin^n x \, dx$, $\int_0^{\frac{\pi}{2}} \cos^n x \, dx$ and $\int_0^{\frac{\pi}{2}} \sin^m x \cos^n x \, dx$ for solving problems where m and n are positive integers.
- Applications of integration.
- Simple problem on evaluation of area bounded by a curve and axes.
- Calculation of Volume of solid formed by revolution of an area about axes(Simple problems).

INTRODUCTION

Integral calculus can be studied under two parts:

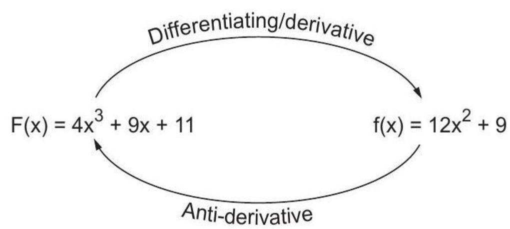
1. Indefinite Integrals
2. Definite Integrals

In this unit, first we will study Integral Calculus which helps in determining functions from their derivatives. Thereafter, we will briefly understand the concept of definite integrals so as to apply them for evaluating area and volume.

Integration as an Inverse Operation of Differentiation

We consider the process of reversing differentiation. That is, we will take a function $f(x)$, and think of all possible functions $F(x)$ which would have $f(x)$ as their derivative. This very thought process leads to the concept of anti-derivative and integration. As by now, you already know how to find out derivatives. So let us consider the function $F(x) = 4x^3 + 9x + 11$, differentiating $F(x)$, we get

$$f(x) = F'(x) = 12x^2 + 9,$$ i.e., figuratively it is



But now the question arises whether the anti-derivative of $f(x)$ as shown above is unique or not? Let us see! These are other functions like $-4x^3 + 9x + 20, 4x^3 + 9x + 200, 4x^3 + 9x$ etc. which have $f(x) = 12x^2 + 9$ as their derivative. This is because the constant term in each of these functions disappears the moment we do differentiation. So, all of these are anti-derivatives of $12x^2 + 9$. In other words, if $F(x)$ is an antiderivative of $f(x)$, then $F(x) + c$ (for any constant c) is also an antiderivative for $f(x)$. Here now comes the definition of Indefinite Integrals.

INDEFINITE INTEGRALS

If f and F are functions of x such that $F'(x) = f(x)$ then the function F is called an antiderivative or primitive or Integral of $f(x)$ with respect to x. Symbolically, it is written as

$$\int f(x)dx = F(x) + c$$

$$\Leftrightarrow \frac{d}{dx}[F(x) + c] = f(x),$$

Where c is called the constant of integration, and $f(x)$ is called the integrand.

Note: $\int f(x)dx = F(x) + c$, represents a family of curves. Here different values of c correspond to different members of this family and these members can be obtained by moving any one of the curves parallel to itself. Furthermore, if we take intersection of a line $x = a$ with the curves, the tangents to the curves at these points of intersection are parallel. This is geometrical interpretation of indefinite integral.

Properties of Indefinite Integrals

1. $\int af(x)dx = a\int f(x)dx$ ('a' is constant).
2. Integral of sum is equal to sum of integrals $\int (f(x) + g(x))dx = \int f(x)dx + \int g(x)dx$.
3. If $\int f(y)dy = F(y) + c$, then $\int f(ax + b)dx = \frac{1}{a}F(ax + b) + c, a \neq 0$.

Standard Results

1. $\int (ax + b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c; n \neq -1$
2. $\int \frac{dx}{ax+b} = \frac{1}{a}\log|ax + b| + c$
3. $\int e^{ax+b}dx = \frac{1}{a}e^{ax+b} + c$
4. $\int a^{px+q}dx = \frac{1}{p}\frac{a^{px+q}}{\log a} + c, (a > 0)$

5. $\int \sin(ax+b)dx = -\frac{1}{a}\cos(ax+b) + c$

6. $\int \cos(ax+b)dx = \frac{1}{a}\sin(ax+b) + c$

7. $\int \tan(ax+b)dx = \frac{1}{a}\log|\sec(ax+b)| + c$

8. $\int \cot(ax+b)dx = \frac{1}{a}\log|\sin(ax+b)| + c$

9. $\int \sec^2(ax+b)dx = \frac{1}{a}\tan(ax+b) + c$

10. $\int \text{cosec}^2(ax+b)dx = -\frac{1}{a}\cot(ax+b) + c$

11. $\int \text{cosec}(ax+b)\cdot\cot(ax+b)dx = -\frac{1}{a}\text{cosec}(ax+b) + c$

12. $\int \sec(ax+b)\cdot\tan(ax+b)dx = \frac{1}{a}\sec(ax+b) + c$

13. $\int \sec x dx = \log|\sec x + \tan x| + c = \log\left|\tan\left(\frac{\pi}{2}+\frac{x}{2}\right)\right| + c$

14. $\int \text{cosec} x dx = \log|\text{cosec} x - \cot x| + c = \log\left|\tan\frac{x}{2}\right| + c$

15. $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + c$

16. $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$

17. $\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{x}{a} + c$

18. $\int \frac{dx}{\sqrt{x^2+a^2}} = \log\left[x+\sqrt{x^2+a^2}\right] + c$

19. $\int \frac{dx}{\sqrt{x^2-a^2}} = \log\left[x+\sqrt{x^2-a^2}\right] + c$

20. $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + c$

21. $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c$

22. $\int \sqrt{a^2-x^2}dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c$

23. $\int \sqrt{x^2+a^2}dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left(x+\sqrt{x^2+a^2}\right) + c$

24. $\int \sqrt{x^2-a^2}dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left(x+\sqrt{x^2-a^2}\right) + c$

25. $\int e^{ax}\cdot\sin bx dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + c$

26. $\int e^{ax}\cdot\cos bx dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) + c$

Example 1. Evaluate $\text{I} = 2\int e^{2x}(\cos 2x - \sin 2x)dx$.

Solution: Here $2e^{2x}(\cos 2x - \sin 2x)$ is the derivative of $e^{2x}\cos 2x$

$$\Rightarrow \; \mathrm{I} = e^{2x}\cos 2x + c$$

Example 2. Evaluate $\mathrm{I} = \int \frac{2dx}{\sin^2 x\cdot\cos^2 x}$.
Solution: Transform the integrand in the following way.

$$\frac{1}{\sin^2 x\cos^2 x} = \frac{\sin^2 x + \cos^2 x}{\sin^2 x\cos^2 x} = \sec^2 x + \operatorname{cosec}^2 x = \frac{d}{dx}[\tan x - \cot x]$$

Hence, $\mathrm{I} = 2\int (\sec^2 x + \operatorname{cosec}^2 x)dx = 2\tan x - 2\cot x + c$

IMPORTANT TECHNIQUES/METHODS OF INTEGRATION

If the integrand is not a derivative of a known function, then the corresponding integrals cannot be found directly. In order to find the integrals in such problems, following main three rules of integration are used:
Rule 1: Integration by substitution (i.e., by changing variables)
Rule 2: Integration by parts.
Rule 3: Integration by partial fractions.

Rule 1: Integration By Substitution (i.e., by Changing Variables)

If $f(x)$ is a continuous differentiable function, then to evaluate integrals of the form $\int \phi(f(x))f'(x)dx$, we substitute $f(x) = t$ and $f'(x)dx = dt$

Hence $I = \int \phi(f(x))f'(x)dx = \int \phi(t)dt$, which can further be integrated easily. Likewise, in integrals of the type $\int [Q(x)]^n Q'(x)dx$ or $\int \frac{Q'(x)}{\sqrt{Q(x)}}dx$ or $\int \frac{Q'(x)}{[Q(x)]^n}dx$, we put $Q(x) = t$ and solve.

Note: In method of substitution, it is important to observe the question minutely and then make a substitution for a function whose derivative also occurs in the integrand.

Examples

Example 3. Evaluate $\int \frac{\cos(\log x)}{x}dx$.
Solution: Let $\log x = t$[as derivative of $\log x$
Then $dt = \frac{1}{x}dx$, substituting it in given integral, we get

$$\mathrm{I} = \int \cos t dt = \sin t + c$$
$$\mathrm{I} = \sin(\log x) + c$$

Example 4. Evaluate $\int \frac{(1+\log x)^3}{x} dx$.

Solution: Let $I = \int \frac{(1+\log x)^3}{x} dx$

Substitute $(1 + \log x) = z$, we get $\frac{1}{x} dx = dz$

$$\Rightarrow \text{ I} = \int z^3 dz = \frac{z^4}{4} + c, \text{ as } z = 1 + \log x$$

$$\Rightarrow \text{ I} = \frac{(1 + \log x)^4}{4} + c$$

Example 5. Evaluate $\int e^x \cos e^x dx$.

Solution: $\int e^x \cos e^x dx = \int \cos e^x . e^x \, dx.$

Put $e^x = \text{t}$

Then

$$\text{I} = \int \cos t dt$$

$$\Rightarrow \text{ I} = \sin t + c$$

$$\Rightarrow \text{ I} = \sin e^x + c$$

Rule 2. Integration by Parts

If u and v are differentiable functions of x then,

$$\int u \cdot v dx = u \int v dx - \int \left[u' \cdot \int v dx\right] dx \qquad (1)$$

where $u' = \frac{du}{dx}$

The given integral in (1) must be separated into two parts, one part (first function) being u and the other part (second function) being v [for this reason it is called integration by parts].

We usually follow the rules as below for integrating with parts:

(i) Choose u and v such that $\int vdx$ and $\int [u' \int vdx]dx$ are simple to integrate.

(ii) If in the integrand only one function is there, then we take unity 1 as the second function. For instance ,in $\int \sin^{-1} x dx, \sin^{-1} x$ is taken as the first function (u) and 1 as the second function (v).

(iii) Generally, we choose first function (u) as the function which comes first in the word ILATE, where I Stands for Inverse function; L Stands for Logarithmic function; A Stands for Algebraic function; T Stands for Trigonometric function; E Stands for Exponential function. For example, in $\int x^2 \cos x dx, x^2$ is taken as the first function (u) and $\cos x$ is taken as the second function (v).

Examples:

Example 6. Evaluate $\int \sec^3 \theta d\theta$

Solution: Let

$$\mathrm{I} = \int \sec\theta \cdot \sec^2\theta d\theta$$

$$\Rightarrow \quad \mathrm{I} = \sec\theta \int \sec^2\theta d\theta - \int \tan\theta(\sec\theta\tan\theta)d\theta$$

$$\Rightarrow \quad \mathrm{I} = \sec\theta\tan\theta - \int (\sec^2\theta - 1)\sec\theta d\theta$$

$$\Rightarrow \quad \mathrm{I} = \sec\theta\tan\theta - \int \sec^3\theta d\theta + \log|\sec\theta + \tan\theta| + c_1$$

$$\Rightarrow \quad 2\mathrm{I} = \sec\theta\tan\theta + \log|\sec\theta + \tan\theta| + c_1$$

$$\Rightarrow \quad \mathrm{I} = \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\log|\sec\theta + \tan\theta| + c\left(c = \frac{c_1}{2}\right)$$

Example 7. Evaluate $\int \log x dx$.

Solution: As only one function is there in the integrand we take unity (1) as the second function

$$\mathrm{I} = \int 1 \cdot \log x dx$$

Then

$$\mathrm{I} = \log x \int 1dx - \int \left[\left[\frac{d}{dx}(\log x)\right] \times \int 1 \cdot dx\right] dx + c$$

$$\Rightarrow \mathrm{I} = x\log x - \int \frac{1}{x} \cdot xdx + c \Rightarrow I = x\log x - x + c$$

Example 8. Evaluate $\int e^x \sin x dx$.

Solution: Let

$$\mathrm{I} = \int e^x \sin x dx$$

Integrating by parts and taking $\sin x$ as first function and e^x as second function we get

$$\mathrm{I} = \sin x \int e^x dx - \int e^x(\cos x)dx + c$$

I = $e^x \sin x$ -[$e^x \cos x - \int e^x(-\sin x)dx] + c$ (again integrating by parts)

I = $e^x \sin x$ - $e^x \cos x$ - $e^x \sin x \, dx + c$ (as I = $\int e^x \sin x dx$)

2I = $e^x \sin x$ - $e^x \cos x$ + c

I = $\frac{1}{2}$ [$e^x \sin x$ - $e^x \cos x$ + c]

Rule 3: Integration By Partial Fractions

A function of the form $\frac{f(x)}{g(x)}$, where $f(x)$ and $g(x)$ are polynomials, is called a rational fraction. It is called proper rational fraction if the degree of $f(x)$ is less than the degree of $g(x)$, otherwise is called an improper rational fraction. An improper rational fraction can further be expressed as the sum of a polynomial and a proper rational fraction (long division process can be used for it). Every proper rational fraction can be expressed as a sum of simpler fractions, called partial fractions. Here, we restrict our study to linear factors in the denominators of partial fraction and hence the cases which arise are listed in the table 2.1, where A_1, A_2, A_3 are constants to be determined accordingly.

Table

Sr. No.	Proper Rational Fraction	Partial Fraction
1.	$\frac{px+q}{(ax+b)(cx+d)}$	$\frac{A_1}{ax+b}+\frac{A_2}{cx+d}$
2.	$\frac{px+q}{(ax+b)^2(cx+d)}$	$\frac{A_1}{ax+b}+\frac{A_2}{(ax+b)^2}+\frac{A_3}{(cx+d)}$
3.	$\frac{px^2+qx+r}{(ax+b)(cx+d)(ex+f)}$	$\frac{A_1}{ax+b}+\frac{A_2}{cx+d}+\frac{A_3}{ex+f}$
4.	$\frac{px^2+qx+r}{(ax+b)^2(cx+d)}$	$\frac{A_1}{(ax+b)}+\frac{A_2}{(ax+b)^2}+\frac{A_3}{(cx+d)}$

Recall: A polynomial in x is a function of the form $a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$ where a_i 's are constants for all $i, a_0 \neq 0$ and n is a positive integer including zero.

Examples:

Example 9. $I = \int \frac{(3x+1)}{(x+1)(x-2)} dx$.

Solution: Here the integrand is a proper rational fraction. So simplifying it further into partial fractions,

we get

$$\frac{3x+1}{(x+1)(x-2)} = \frac{A_1}{x+1} + \frac{A_2}{x-2} \qquad (1)$$

$$\Rightarrow \quad (3x+1) = A_1(x-2) + A_2(x+1)$$

Now, putting $x = 2$ in (1) we get

$$7 = 3A_2 \Rightarrow A_2 = \frac{7}{3} \text{ and putting } x = -1 \text{ gives } -2 = -3A_1 \Rightarrow A_1 = \frac{2}{3}$$

$$\therefore \quad I = \frac{2}{3}\int \frac{dx}{x+1} + \frac{7}{3}\int \frac{dx}{x-2}$$

$$\Rightarrow \quad I = \frac{2}{3}\log|x+1| + \frac{7}{3}\log|x-2| + c \text{ Ans.}$$

Example 10. Evaluate $\int \frac{(3x-4)}{(x-1)^2(x+1)} dx$.

Solution: Let

$$I = \int \frac{(3x-4)}{(x-1)^2(x+1)} dx$$

Integrand is a proper rational fraction and hence simplifying it further using partial fractions we get

$$\frac{3x-4}{(x-1)^2(x+1)} = \frac{A_1}{(x-1)} + \frac{A_2}{(x-1)^2} + \frac{A_3}{x+1}$$

Putting $x = 1$ in (1) we get

$$-1 = 2A_2 \Rightarrow A_2 = \frac{-1}{2}$$

Again put $x = -1$, in (1) we get

$$-7 = 4A_3 \Rightarrow A_3 = -\frac{7}{4}$$

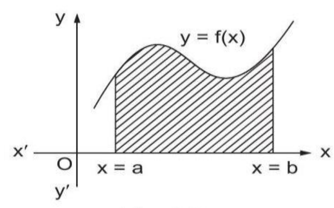
Now equating coefficient of constant terms on both sides of (1) we get

$$-4 = -A_1 + A_2 + A_3$$
$$A_1 = 4 + A_2 + A_3$$
$$\Rightarrow A_1 = 4 - \frac{1}{2} - \frac{7}{4}$$
$$\Rightarrow A_1 = \frac{7}{4}$$

$$\therefore \quad \text{I} = \frac{7}{4}\int \frac{dx}{x-1} - \frac{1}{2}\int \frac{dx}{(x-1)^2} \frac{-7}{4}\int \frac{dx}{x+1}$$
$$\Rightarrow \text{I} = \frac{7}{4}\log|x-1| + \frac{1}{2(x-1)} - \frac{7}{4}\log|x+1| + c$$
$$\text{I} = \frac{7}{4}\log\left|\frac{x-1}{x+1}\right| + \frac{1}{2(x-1)} + c \quad \left(\text{as } \log\frac{m}{n} = \log m - \log n\right)$$

DEFINITE INTEGRALS

The definite integral of a continuous function $f(x)$ defined on the closed interval $[a, b]$, is given by $\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$; where $F(x)$ is the anti-derivative of f, ' a ' is called the lower limit of the integral and ' b ' is called the upper limit of the integral. The definite integral has a unique value.



Geometrically, the definite integral $\int_a^b f(x)dx$ represents the

algebraic area bounded by the curve $y = f(x)$, the ordinates $x = a, x = b$ and the x-axis.

Remarks:

1. $\int_a^b f(x)dx = 0 \Rightarrow$ the equation $f(x) = 0$ has atleast one root lying in (a, b) provided f is a continuous function in (a, b)
2. $\int_a^a f(x)dx = 0$ (i.e., when lower limit = upper limit)

Example 11. Evaluate:

(i) $\int_0^2 x^3 dx$

(ii) $\int_0^{\pi/4} \sin^4 2t \cos 2t dt$

Solution:

(i) Let $\text{I} = \int_0^2 x^3 dx$

As we know that, $\int x^3 dx = \frac{x^4}{4} = F(x)$

$\therefore\ \mathrm{I} = F(2) - F(0) = \frac{16}{4} - 0 = 4$

(ii) Let $\mathrm{I} = \int_0^{\pi/4} \sin^4 2t \cos 2t dt$

Then $\int \sin^4 2t \cos 2t dt = \frac{1}{2}\int z^4 \cdot dz$ [Put $\sin 2t = z \Rightarrow 2\cos 2t dt = dz$]

$$= \frac{z^5}{10} = \frac{(\sin 2t)^5}{10} = F(t)$$

Then $\mathrm{I} = F\left(\frac{\pi}{4}\right) - F(0) = \frac{1}{10}\left[\sin\frac{\pi}{2} - \sin 0\right]$

$\Rightarrow\ \mathrm{I} = \frac{1}{10}$

Some Common Properties of Definite Integrals

1. $\int_a^b f(x)dx = \int_a^b f(t)dt$ provided f is same.
2. $\int_a^b f(x)dx = -\int_b^a f(x)dx$.
3. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$, where c may lie inside or outside the interval [a, b]. This property is to be used when f is piecewise continuous in (a, b).
4. $\int_{-a}^{a} f(x)dx = \int_0^a [f(x) + f(-x)]dx = \begin{cases} 0; & \text{if } f(x) \text{ is an odd function} \\ 2\int_0^a f(x)dx; & \text{if } f(x) \text{ is an even function} \end{cases}$
5. $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$, in particular $\int_0^a f(x)dx = \int_0^a f(a-x)dx$
6. $\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx = \begin{cases} 2\int_0^a f(x)dx; & \text{if } f(2a-x) = f(x) \\ 0; & \text{if } f(2a-x) = -f(x) \end{cases}$
7. $\int_0^{nT} f(x)dx = n\int_0^T f(x)dx, (n \in I)$; where ' T ' is the period of the function i.e., $f(T + x) = f(x)$

Note: $\int_x^{T+x} f(t)dt$ will be independent of x and equal to $\int_0^T f(t)dt$

Examples:

Example 12. Evaluate $\int_0^2 f(x)dx$, if $f(x) = \begin{cases} x^3 & 0 < x < 1 \\ 2x+1 & 1 \le x \le 2 \end{cases}$

Solution:

$$\int_0^2 f(x)dx = \int_0^1 f(x)dx + \int_1^2 f(x)dx$$
$$= \int_0^1 x^3 dx + \int_1^2 (2x+1)dx$$
$$= \left[\frac{x^4}{4}\right]_0^1 + [x^2+x]_1^2 = \frac{1}{4} + [4+2-2] = \frac{1}{4} + 4 = \frac{17}{4}$$

Example 13. Evaluate $\mathrm{I} = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}} dx$.

Solution: Let

$$\mathrm{I} = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}} \cdot dx \qquad (1)$$

$$\Rightarrow \mathrm{I} = \int_0^{\pi/2} \frac{\sqrt{\cos\left(\frac{\pi}{2}-x\right)}}{\sqrt{\cos\left(\frac{\pi}{2}-x\right)}+\sqrt{\sin\left(\frac{\pi}{2}-x\right)}} dx$$

$$\left(\text{Using property of definite integral } \int_0^a f(x)dx = \int_0^a f(a-x)dx\right)$$

$$\mathrm{I} = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}} \cdot dx \qquad (2)$$

From (1) and (2) we get

$$2\mathrm{I} = \int_0^{\pi/2} \frac{(\sqrt{\sin x}+\sqrt{\cos x})}{(\sqrt{\sin x}+\sqrt{\cos x})} \cdot dx \Rightarrow \mathrm{I} = \frac{1}{2}\int_0^{\pi/2} dx$$
$$\Rightarrow \mathrm{I} = \frac{1}{2}[x]_0^{\pi/2} = \frac{1}{2}\left[\frac{\pi}{2}\right] \Rightarrow \mathrm{I} = \frac{\pi}{4}$$

USE OF WALLI'S INTEGRAL FORMULA $\left(\int_0^{\pi/2} \sin^m x \cos^n x dx\right)$

This formula is attributed to the English Mathematician John Wallis (1616-1703). Its proof is based on reduction formulae. We will study following two cases:

Case I. $\int_0^{\pi/2} \sin^m x\cos^n xdx$. The value of this definite integral with trigonometric integrand can be found easily using Walli's integral formula as given below:
If m, n are both positive integers, then

$$\int_0^{\pi/2} \sin^m x\cos^n xdx = \int_0^{\pi/2} \sin^n x\cos^m xdx$$
$$\begin{cases} = \dfrac{(m-1)(m-3)\ldots(1 \text{ or } 2)\cdot(n-1)(n-3)\ldots(1 \text{ or } 2)}{(m+n)(m+n-2)\ldots(1 \text{ or } 2)}\cdot\dfrac{\pi}{2}, \\ = \dfrac{(m-1)(m-3)\ldots(1 \text{ or } 2)\cdot(n-1)(n-3)\ldots(1 \text{ or } 2)}{(m+n)(m+n-2)\ldots(1 \text{ or } 2)}\cdot 1, \end{cases} \text{Otherwise } n \text{ both are even integer}$$

Case II. If n is a positive integer, then

$$\int_0^{\pi/2} \sin^n xdx = \int_0^{\pi/2} \cos^n xdx = \begin{cases} \dfrac{(n-1)(n-3)(n-5)\ldots..3\cdot 1}{n(n-2)(n-4)\ldots 4\cdot 2}\cdot\dfrac{\pi}{2}; & \text{if } n \text{ is even} \\ \dfrac{(n-1)(n-3)(n-5)\ldots.4\cdot 2}{n(n-2)(n-4)\ldots 5\cdot 3}\cdot 1; & \text{if } n \text{ is odd} \end{cases}$$

This is special case of Walli's integral formula.
Example 14. Evaluate $\int_0^{\pi/2} \sin^6 x\cos^2 xdx$.
Solution: Using Walli's integral formula we get

$$\int_0^{\pi/2} \sin^6 x\cos^2 xdx = \frac{(6-1)(6-3)(6-5)\cdot(2-1)}{(6+2)(6+2-2)(6+2-4)(6+2-6)}\cdot\frac{\pi}{2}$$
$$= \frac{5\cdot 3\cdot 1\cdot 1}{8\cdot 6\cdot 4\cdot 2}\cdot\frac{\pi}{2} = \frac{15}{384}\cdot\frac{\pi}{2} = \frac{15}{768}\cdot\pi$$

Examples 15. Evaluate $\mathrm{I} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^4 x \cos^6 xdx$

Solution: $\mathrm{I} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^4 x \cos^6 xdx$

$\Rightarrow \mathrm{I} = 2\int_0^{\pi/2} \sin^4 x\cos^6 xdx$ (as the integrand is an even function)

$\Rightarrow \mathrm{I} = \frac{2\cdot(3\cdot 1)(5\cdot 3\cdot 1)}{10\cdot 8\cdot 6\cdot 4\cdot 2}\cdot\frac{\pi}{2} = \frac{3\pi}{256}$

Examples 16. Evaluate $\mathrm{I} = \int_0^{\pi/2} \cos^7 xdx$.

Solution: Using Wallis formula, as the exponent is odd, so we get

$$\mathrm{I} = \frac{6 \times 4 \times 2}{7 \times 5 \times 3} = \frac{16}{35}$$

Examples 17. $\mathrm{I} = \int_0^{\pi/2} \sin^8 x dx$.

Solution: Using Walli's formula and as the exponent is even in I we get,

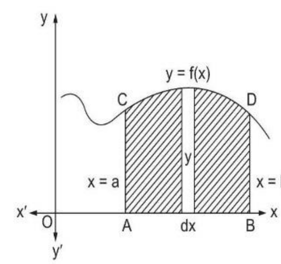
$$\mathrm{I} = \frac{7 \times 5 \times 3 \times 1}{8 \times 6 \times 4 \times 2} \times \frac{\pi}{2} \Rightarrow \mathrm{I} = \frac{35\pi}{256}$$

APPLICATIONS OF INTEGRATION

In this section, we would learn to deal with simple problems related to evaluation of area bounded by a curve and axes. Apart from this we would learn to calculate the volume of a solid formed by revolution of an area about axes (simple problems only).

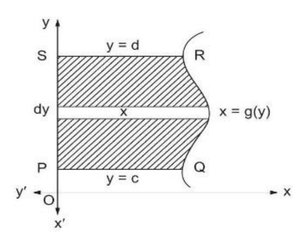
I. Area Bounded by a Curve and Axes

We learnt that definite integrals help us in finding area. When we find area between a curve and axes the following two cases arise:



Case I. When we are finding the area A bounded by the curve $y = f(x)$, x-axis and the ordinates $x = a$ and $x = b$ as shown in fig. 2.3.
In this case we have to find out the area enclosed in $ABDCA$. This area is considered as being made up of many very-thin vertical strips /rectangles. We consider one arbitrary strip of height y and width dx. For this strip area $dA = ydx$. (here $y = f(x)$). This area is called an elementary area. We sum up all such elementary areas by taking integration from $x = a$ to $x = b$ (in $ABDCA$) and get the total area bounded by the curve $y = f(x)$, x-axis and the lines $x = a, x = b$ as below:

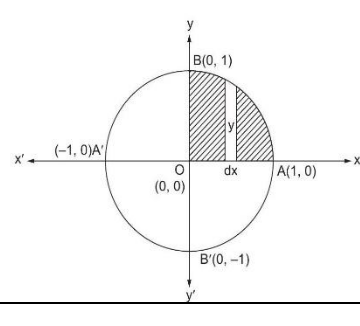
$$A = \int_a^b dA = \int_a^b ydx = \int_a^b f(x)dx \text{ i.e., } A = \int_a^b f(x)dx$$



Case II: When we are finding the area A bounded by the curve $x = g(y)$, y-axis and the lines $y = c, y = d$, as shown in (Fig. 2.4). In this case we have to find out area enclosed in PQRSP. Here, we consider area PQRSP as made up of very thin horizontal strips/rectangles of length x and width dy.

$$\therefore \text{ Area } A = \int_c^d g(y)dy$$

Example 18. Evaluate the area enclosed by one quadrant of the circle $x^2 + y^2 = 1$. Hence find the total area enclosed by the given circle.



Solution: The circle under consideration is $x^2 + y^2 = 1$

As circle is symmetrical along both the axes and points of intersection of (1) with x-axis are $A(1,0)$ and $A'(-1,0)$ and with y-axis are $B(0,1)$ and $B'(0,-1)$ we get Fig. 2.6. As area of all quadrants is equal we find the area enclosed by quadrant $OABO$.

$$\text{i.e., } Q_1 = \int_0^1 y dx$$

By taking vertical strip

$$Q_1 = \int_0^1 \sqrt{1-x^2}dx$$

(as $y = \pm\sqrt{1-x^2}$ taking the positive sign as $OABO$ lies in first quadrant from (1)

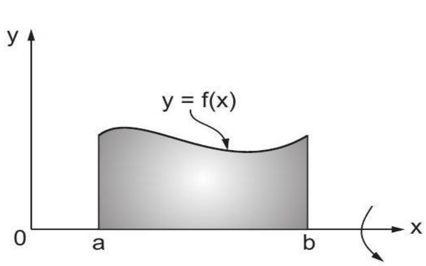
$$\Rightarrow \quad Q_1 = \left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}\frac{x}{1}\right]_0^1$$
$$\Rightarrow \quad Q_1 = \left[\frac{1}{2}\sin^{-1}1\right]$$
$$\Rightarrow \quad Q_1 = \frac{\pi}{4} \text{ Ans.}$$

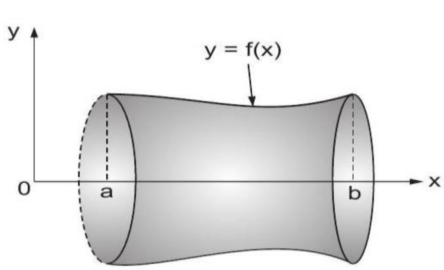
Again, area A enclosed by the circle $= 4 \times Q_1$

$$\Rightarrow \quad A = 4 \times \frac{\pi}{4}$$
$$\Rightarrow \quad A = \pi \text{ sq. units}$$

II. Volume of a Solid formed by Revolution of an Area about Axes (Using Disk Method)

One of the important and simplest applications of integration is to find the volumes of solid formed by revolution. Following two cases arise:



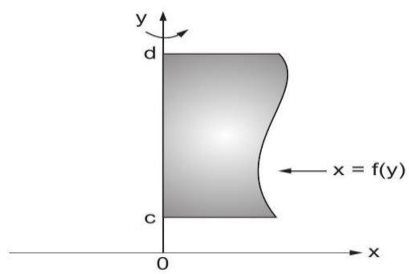


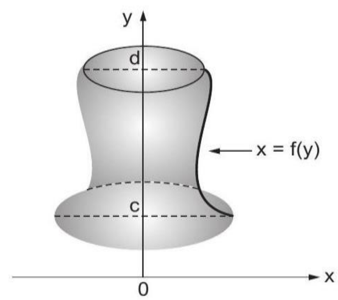
Case I: Revolution about X-axis: The volume of a solid formed by revolution of an area about X-axis, bounded by the curve $y = f(x)$, the ordinates $x = a, x = b$ and the X-axis is (Shown on the above fig.)

$$V = \int_a^b \pi y^2 dx$$

Case II: Revolution about Y-axis: The volume of a solid formed by revolution of an area about Y-axis, bounded by the curve $x = f(y)$, the lines $y = c, y = d$ and the Y-axis is

$$V = \int_c^d \pi x^2 dy$$





Example 19. Let $y = f(x) = x^2$ be the curve given on the interval [0,2]. Find the volume of solid generated by revolving the region between the curve in the given interval around X-axis.
Solution: We know that volume of solid of revolution around X-axis in $[a, b]$ is given by (for $y = f(x)$)

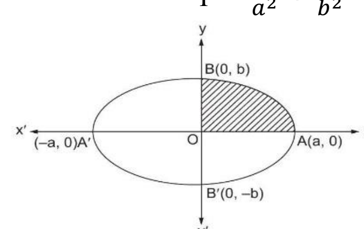
$$V = \int_a^b \pi y^2 dx \text{ i.e., } V = \int_0^2 \pi (x^2)^2 dx \Rightarrow V = \pi \int_0^2 x^4 dx \Rightarrow V = \pi \left| \frac{x^5}{5} \right|_0^2 \Rightarrow V = \frac{32\pi}{5}$$

Example 20. Let $y = f(x) = x^2$ be given on the interval [0,2]. Find the volume of solid generated by revolving the region between the curves (in the given interval $[0,2]$) around Y-axis.
Solution: As here we have to find the revolution around y-axis,

$\therefore \quad V = \int_a^b \pi x^2 dy$ formula will be used. Here $y = x^2 \Rightarrow x^2 = y$ on $[0,2]$

$$\therefore \quad V = \pi \int_0^2 y dy \Rightarrow V = \pi \left[\frac{y^2}{2} \right]_0^2 \Rightarrow V = 2\pi$$

Example 21. Evaluate the volume of the solid generated by the revolution of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ about its minor axis .
Solution: Minor axis of the ellipse



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 (b < a) \text{ is the } Y\text{-axis}$$

∴ We use the formula for revolution around Y-axis i.e.,

$$V = \int_a^b \pi x^2 dy \text{ on interval } [a, b]$$

Now, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow x^2 = \frac{a^2}{b^2}(b^2 - y^2)$
∴ Volume of the ellipse about its minor axis (Y-axis)
$=$ volume generated by revolution of arc $B'AB$ about Y-axis (As shown in the fig.)
$=$ twice the volume generated by revolution of the arc BA about Y-axis (as ellipse is symmetrical about X-axis). But the arc BA varies from $y = 0$ to $y = b$

$$\therefore \text{ Required volume } = 2\int_0^b \pi x^2 dy$$

$$\Rightarrow V = 2\pi \int_0^b \frac{a^2}{b^2}(b^2 - y^2) dy \Rightarrow V = 2\pi \frac{a^2}{b^2} \int_0^b (b^2 - y^2) dy$$

$$\Rightarrow V = \frac{2\pi a^2}{b^2}\left[b^2 y - \frac{y^3}{3}\right]_0^b \Rightarrow V = \frac{2\pi a^2}{b^2}\left[b^3 - \frac{b^3}{3}\right] \Rightarrow V = \frac{4\pi a^2 b}{3}$$

POSSIBLE SHORT TYPE QUESTIONS WITH ANSWER

Q.1 Evaluate $\mathrm{I} = \int \frac{2dx}{\sin^2 x \cdot \cos^2 x}$.

Solution: Transform the integrand in the following way.

$$\frac{1}{\sin^2 x \cos^2 x} = \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x} = \sec^2 x + \operatorname{cosec}^2 x = \frac{d}{dx}[\tan x - \cot x]$$

Hence, $\mathrm{I} = 2\int (\sec^2 x + \operatorname{cosec}^2 x)dx = 2\tan x - 2\cot x + c$

Q.2 Evaluate $\int \frac{\cos(\log x)}{x}dx$.

Solution: Let $\log x = t$[as derivative of $\log x$

Then $dt = \frac{1}{x}dx$, substituting it in given integral, we get

$$\mathrm{I} = \int \cos t dt = \sin t + c$$

$$\mathrm{I} = \sin(\log x) + c$$

Q.3 Evaluate $\int \frac{(1+\log x)^3}{x}dx$.

Solution: Let $I = \int \frac{(1+\log x)^3}{x}dx$

Substitute $(1 + \log x) = z$, we get $\frac{1}{x}dx = dz$

$$\Rightarrow \mathrm{I} = \int z^3 dz = \frac{z^4}{4} + c, \text{ as } z = 1 + \log x$$

$$\Rightarrow \mathrm{I} = \frac{(1 + \log x)^4}{4} + c$$

Q.4 Evaluate $\mathrm{I} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^4 x \cos^6 x dx$.

Solution: $\mathrm{I} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^4 x \cos^6 x dx$

$\Rightarrow \mathrm{I} = 2\int_0^{\pi/2} \sin^4 x \cos^6 x dx$ (as the integrand is an even function)

$\Rightarrow \mathrm{I} = \frac{2\cdot(3\cdot 1)(5\cdot 3\cdot 1)}{10\cdot 8\cdot 6\cdot 4\cdot 2}\cdot\frac{\pi}{2} = \frac{3\pi}{256}$

Q.5 Let $y = f(x) = x^2$ be the curve given on the interval [0,2]. Find the volume of solid generated by revolving the region between the curve in the given interval around X-axis.

Solution: We know that volume of solid of revolution around X-axis in $[a,b]$ is given by (for $y = f(x)$)

$$V = \int_a^b \pi y^2 dx \text{ i.e., } V = \int_0^2 \pi (x^2)^2 dx \Rightarrow V = \pi \int_0^2 x^4 dx \Rightarrow V = \pi \left| \frac{x^5}{5} \right|_0^2 \Rightarrow V = \frac{32\pi}{5}$$

Q.6 Let $y = f(x) = x^2$ be given on the interval [0,2]. Find the volume of solid generated by revolving the region between the curves (in the given interval $[0,2]$) around Y-axis.
Solution: As here we have to find the revolution around y-axis,

$$\therefore \quad V = \int_a^b \pi x^2 dy \text{ formula will be used. Here } y = x^2 \Rightarrow x^2 = y \text{ on } [0,2]$$

$$\therefore \quad V = \pi \int_0^2 y dy \Rightarrow V = \pi \left[\frac{y^2}{2} \right]_0^2 \Rightarrow V = 2\pi$$

POSSIBLE LONG TYPE QUESTIONS

Q.1 Solve $\int \sqrt{\frac{1-x}{1+x}}$ dx.

Q.2 Evaluate $\int \frac{(3x-4)}{(x-1)^2(x+1)} dx$.

Q.3 Evaluate $\text{I} = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}} dx$.

Q.4 Evaluate $\int e^x \sin x dx$.

Q.5 Prove that $\int_0^{\frac{\pi}{4}} \log(1 + tanx)dx = \frac{\pi}{8} \log 2$.

UNIT–III

CO-ORDINATE GEOMETRY

Learning Objective:

a)Equation of straight line in various standard forms (without proof), inter section of two straight lines,

b) Angle between two lines. Parallel and perpendicular lines, perpendicular distance formula.

c)General equation of a circle and its characteristics. To find the equation of a circle, given:

i. Centre and radius, ii. Three points lying on it and iii. Coordinates of end points of a diameter;

d) Definition of conics (Parabola, Ellipse, Hyperbola) their standard equations without proof. Problems on conics when their foci, directories or vertices are given.

Origin of Coordinate Geometry

French mathematician René Descartes first carried out a systematic study of geometry by the use of algebra in his book 'La Géométrie' (published in 1637). Modern coordinate (analytic) geometry is called 'Cartesian' after him.

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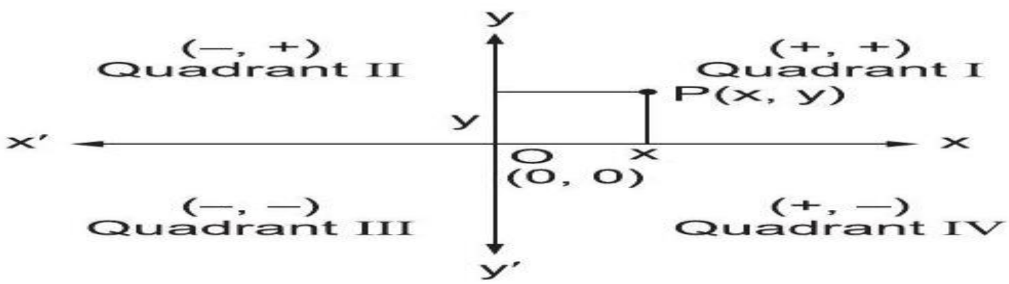
CONCEPT OF COORDINATE GEOMETRY

The mathematical formulations of algebra and geometry are designated as coordinate geometry. For the first time is the history of mathematics the use of algebra and geometry simultaneously was carried out by the then great French Mathematician René Descartes.

The algebraic combination of analysis and geometry is referred to as analytical geometry.

Cartesian Co-ordinates System

In order to have the concept of cartesian coordinates system we consider the geometrical space. Consider an $X-Y$ plane, in which there are two dimensions. One is called x-axis and the other is called y-axis. Since there are two axes (perpendicular) in this geometrical space that's why it is called two-dimensional rectangular coordinate system. The horizontal line is labelled as X-axis and the vertical line as Y-axis. At this stage we make it clear that X-axis is symbolic as independent variable and Y-axis is symbolic as dependent variable. In Fig. 3.1 in the horizontal, X-axis is taken and in the vertical, Y-axis is taken. It is to be noted in the Cartesian system, the coordinates of a point are written as (x, y), in which, first place is specifically meant for X-axis and second place is meant for Y-axis. $O(0,0)$ is the intersection point between X-axis and Y-axis.



STRAIGHT LINE

Consider two vertices (points) on a geometrical space say P and Q. The shortest distance between the two vertices P and Q is called straight line. Any straight line can be viewed as a curve but without curvature.

If P and Q are any two vertices denoted as $P(x_1, y_1)$ and $Q(x_2, y_2)$, then distance between P and Q is,

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

If the two vertices are lying on each other than $PQ = 0$

Example . The number of vertices on X-axis, which are at a distance $a(a < 4)$ from another vertex $Q(3,4)$ is
(1) 2
(2) 3
(3) 4
(4) not defined

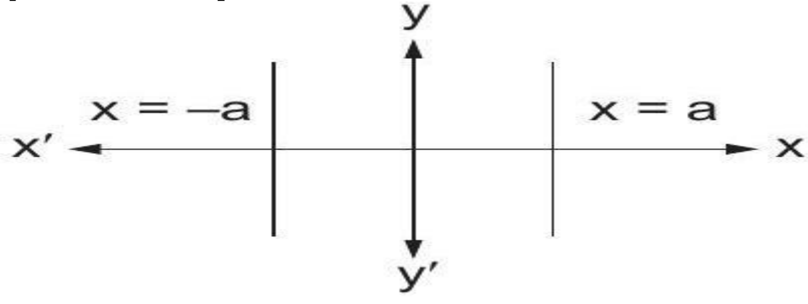
Solution: Let a vertex on x-axis be $(x_1, 0)$ then its distance D from the vertex (3,4) is $a = \sqrt{(x_1 - 3)^2 + (0 - 4)^2}$

$$\Rightarrow \quad a = \sqrt{(x_1 - 3)^2 + 16} \Rightarrow a^2 = (x_1 - 3)^2 + 16 \Rightarrow (x_1 - 3)^2 = a^2 - 16$$
$$\Rightarrow \quad x_1 - 3 = \pm\sqrt{a^2 - 16} \Rightarrow x_1 = 3 \pm \sqrt{a^2 - 16} \Rightarrow x_1 = 3 \pm \sqrt{a^2 - 16}$$

As $a < 4$, therefore, x_1 is not defined. Ans. (4)
Remark: A relation in x and y which is satisfied by co-ordinates of every point on a line is known as equation of straight line. It is given by $ax + by + c = 0$. (a and $b \neq 0$)

Equation of Vertical Lines

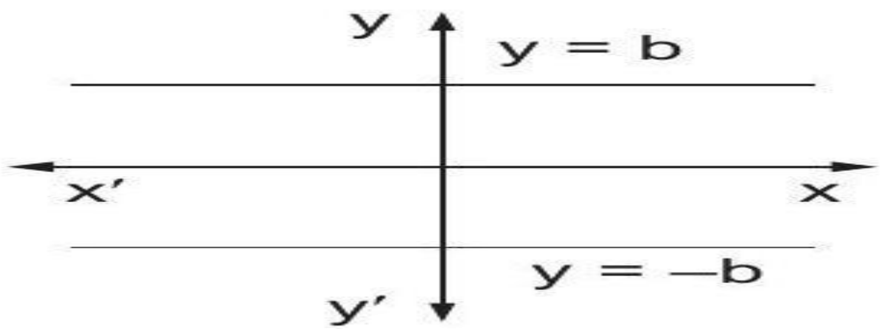
(1) Equation of Y-axis is $x = 0$
(2) Equation of a line parallel to Y-axis at distance of a units is $x = a$ or $x = -a$.



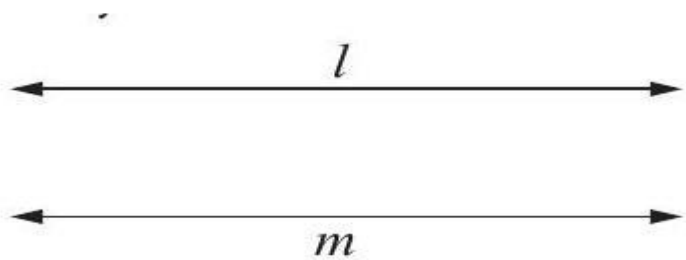
Equation of Horizontal Line

(1) Equation of X-axis is $y = 0$

(2) Equation of a line parallel to X-axis at a distance of ' b ' units is $y = b$ or $y = -b$



Parallel lines: Two lines are said to be parallel if they do not have any intersection vertex this implies that the two lines are disjoint



(l and m are parallel lines)

Note: Equation of line parallel to $ax + by + c = 0$ is $ax + by + \lambda = 0$ (λ-parameter)
Perpendicular lines: Two lines are said to be perpendicular lines if they intersect at 90° angle.

Equation of a line perpendicular to line $ax + by + c = 0$ is $bx - ay + \lambda_1 = 0$ (λ_1 = parameter).
Ex. Co-ordinate axes are perpendicular lines.
Coincident lines: Two lines are said to be coincident on each other if one line overlaps the other lines.

It is to be noted that when two lines (parallel) are moving in the same direction, they will meet at a vertex (point) which is called vertex at infinity. But geometrically it is not possible to construct.

Example . Prove that the lines
(a) $2x + 5y + c = 0; 4x + 10y + 2c_1 = 0$ are parallel to each other.
(b) $3x + 4y + 7 = 0; 4x - 3y + 1 = 0$ are perpendicular to each other

Solution: (a) Let

$$2x + 5y + c = 0 - - - (1)$$
$$4x + 10y + 2c_1 = 0 - - - (2)$$

$$(1) \Rightarrow \quad y = -\frac{2}{5}x - \frac{c}{5}$$

$$(2) \Rightarrow \quad y = -\frac{4}{10}x - \frac{2c_1}{10}$$

$$\Rightarrow \quad y = -\frac{2x}{5}x - \frac{c_1}{5}$$

$$m_2 = \frac{-2}{5} = m_1 \ \therefore \text{(1) and (2) are parallel lines. Ans.}$$

(b) Given lines are

$$3x + 4y + 7 = 0 \text{ and } 4x - 3y + 1 = 0$$

$$\text{i.e.,} \quad y = \frac{-3}{4}x - \frac{7}{4} \text{ and } y = \frac{4}{3}x + \frac{1}{3}$$

$$\text{i.e.,} \quad m_1 = -\frac{3}{4} \text{ and } m_2 = \frac{4}{3}$$

$$\therefore \quad m_1 m_2 = \left(\frac{-3}{4}\right) \times \left(\frac{4}{3}\right) = -1.$$ Hence lines are perpendicular to each other. Ans.

Slope of a Line

Suppose a line PQ makes an angle $\phi(0^\circ \leq \phi < 180^\circ)$ with the right side directions of X-axis, then slope of this line will be designated by $\tan\phi$ and is denoted by symbol m i.e., $m = \tan\phi$. If $P(x_1, y_1)$ and $Q(x_2, y_2)$ are two points such that, $x_1 \neq x_2$, then slope of the line $PQ = \frac{y_2 - y_1}{x_2 - x_1}$

Remark:

(i) If $\phi = 90^\circ$, then slope m does not exist and line is disjoint to Y-axis.
(ii) If $\phi = 0$, then $m = 0$ and the line is disjoint to X-axis.
(iii) (a) If lines are disjoint then $m_1 = m_2$ and vice versa.
(b) If lines are perpendicular to each other, then $m_1 \cdot m_2 = -1$ and vice versa.

Equation of Straight Line in Various Standard Forms

(i) Slope intercept form: Suppose m be the slope of a line and c its intercept on Y-axis. Then the equations of such straight line is given as

$$y = mx + c - - - - - (1)$$

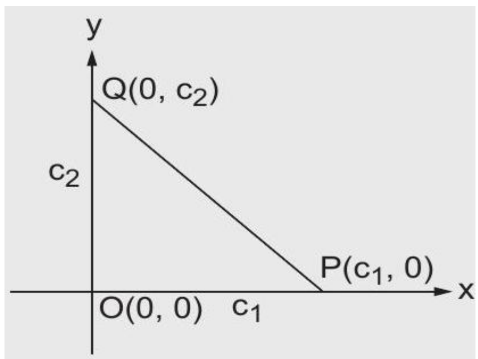
where y and x are dependent and independent variables respectively. If in this equation we put $c = 0$, equation (1) reduces to $y = mx$.

(ii) Point slope from: If m be the slope of a straight line which passes through a point (x_1, y_1), then its equation is written as $y - y_1 = m(x - x_1)$
(iii) Intercept form: Suppose c_1 and c_2 be the two intercepts obtained by a straight line on the X-axis and Y-axis, then its equation is written as $\frac{x}{c_1} + \frac{y}{c_2} = 1$.

Note:

(i) Distance of intercept of line between the coordinate axis $= \sqrt{c_1^2 + c_2^2}$
(ii) Area of triangle $OPQ = \frac{1}{2} PO \cdot QO = \left|\frac{1}{2} c_1 \cdot c_2\right|$
(iii) Two point form: Equations of a line going through two points (x_1, y_1) and (x_2, y_2) is expressed as

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) \text{ or } \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$



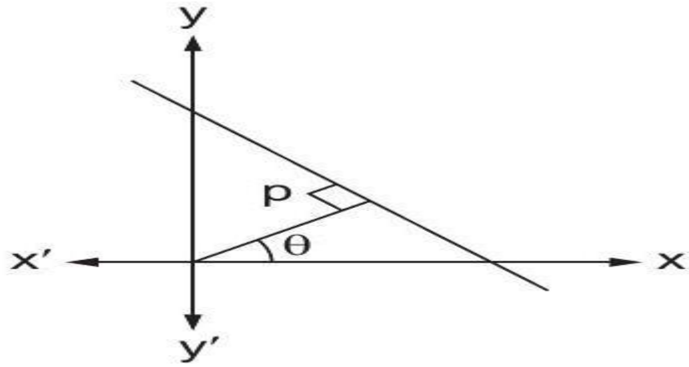
where $|x|$ denotes determinant of x.
Example . Find the equation of line passing through (0,0) and (2,2).
Solution: Using the formula

$$y - y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1), \text{ we get } y - 0 = \left(\frac{2 - 0}{2 - 0}\right)(x - 0)$$

$\Rightarrow 2y = 2x \Rightarrow y = x \ \ (Ans.)$

(iv) Normal form: If p is the distance of perpendicular on a straight line from the origin, and θ is the angle between the positive X-axis and this perpendicular, then the equation of this straight line is expressed as
$x\cos\theta + y\sin\theta = p$ (p is always greater then 0) where $0 \leq \theta \leq 2\pi$
(v) **General form:** Consider two variables x and y and three constants a, b, c. The linear or first degree equation in variables x and y is represented as

$$ax + by + c = 0 - - - - - (1)$$



This is the equation of a straight line. The algebraic structure given in (1) is the general form of straight line. Now we derive the following from (1)

(i) Slope of the line $= \frac{-a}{b} = -\frac{\text{coefficient of } x}{\text{coefficient of } y}$

(ii) Intercept by this line on x-axis $= -\frac{c}{a}$ and also the intercept by this line on y-axis $= -\frac{c}{b}$.

(iii) In order to change the general form of straight line into the normal form we take c to the R.H.S of (1) and convert it to positive, then divide the entire equation by $\sqrt{a^2+b^2}$.

Angle between Two Lines

(a) Angle ϕ between two straight lines (say)

$$y = a_1x + c_1 \text{ and } y = a_2x + c_2, \text{ is}$$
$$\tan\phi = \pm\left(\frac{a_1 - a_2}{1 + a_1a_2}\right)$$

It is to be noted that-

(i) There are two angles generated between two straight lines but usually the acute angle is taken as the angle between two lines. Hence the value of ϕ can be determined on considering only the positive value of $\tan\phi$.

(ii) Let a_1, a_2, a_3 be the slopes of three straight lines $L_1 = 0; L_2 = 0; L_3 = 0$, where $a_1 > a_2 > a_3$ then the interior angles of any triangle found by these formulae are given by ($\triangle EFG$)

$$\tan E = \frac{a_1 - a_2}{1 + a_1a_2}, \tan F = \frac{a_2 - a_3}{1 + a_2a_3}, \tan G = \frac{a_3 - a_1}{1 + a_3a_1}$$

(b) Consider the equation of two straight lines-

$$a_1x + b_1y + c_1 = 0$$
$$a_2x + b_2y + c_2 = 0$$

then these straight lines are

(i) Parallel $\Leftrightarrow \frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$

(ii) Perpendicular $\Leftrightarrow a_1a_2 + b_1b_2 = 0$ ($\because a_1a_2 = -1$)

(iii) Coincident $\Leftrightarrow \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

Remark: Straight line making a given angle with a line - Equations of the straight lines going through a point whose coordinates are (x_1, y_1) and forming an angle (say) α, with the straight line whose linear equation is $y = mx + c$ is written as

$$y - y_1 = \frac{m \pm \tan \alpha}{1 \mp m \tan \alpha}(x - x_1)$$

Example . Which of the following is correct value if $3x + 4y - 5 = 0$ and $2x + ky + 6 = 0$ are two equations of two perpendicular lines

(1) $-\frac{3}{2}$

(2) $\frac{3}{2}$

(3) 3

(4) 4

Solution: Slope of the 1st line $= -\frac{\text{coefficient of } x}{\text{coefficient of } y}$

Consider equation $3x + 4y - 5 = 0$, we have

$$m_1 = \frac{-3}{4}.$$

From equation $2x + ky + 6 = 0$, we have

$$m_2 = \frac{-2}{k}$$

If $m_1 m_2 = -1$, then the two straight lines are perpendicular and we have

$$m_1 = \frac{-3}{4}, m_2 = \frac{-2}{k}$$

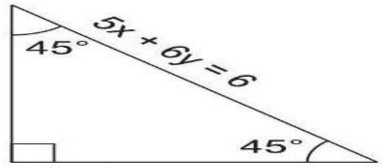
$$\therefore\ m_1 m_2 = -1 \Rightarrow -\frac{-3}{4} \times \frac{-2}{k} = -1$$

$$\frac{3}{2k} = -1 \Rightarrow 3 = -2k \Rightarrow k = \frac{-3}{2}$$

Ans. (1)

Example . Find the equation to the sides of an isosceles right angled triangle, the equations of whose hypotenuse is $5x + 6y = 6$ and the opposite vertex is the point $P(3,3)$.

Solution: Obtain the equation of the straight lines passing through the given point $(3,3)$ and making equal angles of $45°$ with the given straight line $5x + 6y = 6$ or $5x + 6y - 6 = 0$. Slope of the straight line $5x + 6y - 6 = 0$ is given by



$m_1 = -\frac{\text{Coeff. of } x}{\text{Coeff. of } y} = -\frac{5}{6}$

$$\tan 45° = \pm\frac{m - m_1}{1 + mm_1}$$

or

$$1 = \pm\frac{m - m_1}{1 + mm_1} = +\frac{m + \frac{5}{6}}{1 + m\left(\frac{-5}{6}\right)} = +\frac{6m + \frac{5}{6}}{6 - 5\frac{m}{6}} \ (\because \tan 45° = 1)$$

$$\Rightarrow \ 1 = +\frac{6m + 5}{6 - 5m} \Rightarrow 6 - 5m = 6m + 5 \ \text{ or } \ m_A = \frac{1}{11}$$

Now consider
or

$$6 + 5 = -6m + 5m \ \text{ or } 11 = -m \ \text{ or } \ m_B = -11$$

Hence the required equations of two straight lines are

$$y - 3 = m_A(x - 3)$$
$$\Rightarrow \quad y - 3 = \frac{1}{11}(x - 3) \Rightarrow x - 11y + 30 = 0$$
$$\text{and} \quad y - 3 = m_B(x - 3)$$
$$\Rightarrow \quad y - 3 = -11(x - 3) \Rightarrow 11x + y - 36 = 0$$

Distance of Perpendicular from a Point on a Line

Let us consider a straight line whose equation is given as
$ax + by + c = 0$ then, distance of perpendicular from a point $P(x_1, y_1)$ is expressed as $\left|\frac{ax_1+by_1+c}{\sqrt{a^2+b^2}}\right|$
In particular, the distance of the perpendicular from the origin $(0,0)$ lying on the line

$$ax + by + c = 0 \text{ is } p = \frac{|c|}{\sqrt{a^2 + b^2}}, (\text{ since } x = 0, y = 0)$$

Example 6. Verify whether any line can be drawn through the point $(4, -6)$ so that its distance from $(-2,3)$ will be equal to 10 .
Solution: Let the equations on straight line through point $P(4, -6)$ so that its distance from $(-2,3)$ is equal to 10 .
We know that the equation is constructed on taking the point $P(4, -6)$ with slope of m is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

We have $y + 6 = m(x - 4)$
$\Rightarrow \ y + 6 = mx - 4m \Rightarrow mx - y - 4m - 6 = 0$
then $\frac{|m(-2)-3-4m-6|}{\sqrt{m^2+1}} = 10$
$\frac{|-6m-9|}{\sqrt{m^2+1}} = 10 \Rightarrow \frac{6m+9}{\sqrt{m^2+1}} = 10$ or $6m + 9 = 10\sqrt{m^2 + 1}$
Squaring both the sides, we have

$$6m + 9 = 10\sqrt{m^2 + 1}$$
$$(6m + 9)^2 = 100(m^2 + 1)$$
$$\Rightarrow$$
$$\Rightarrow \ 36m^2 + 108m + 81 = 100m^2 + 100$$
$$\Rightarrow 64m^2 - 108m + 19$$

Now the discriminant $b^2 - 4ac$ is

$(108)^2 - 4(64)(19) = 11664 - 4864 = 6800 > 0$ which is positive.

Hence such line is possible.

Distance between Two Parallel Lines

(i) The distance between two disjoint (parallel) lines

$ax + by + c_1 = 0$ and $ax + by + c_2 = 0$ is determined as $D = \dfrac{|c_1 - c_2|}{\sqrt{a^2 + b^2}}$

It is to be noted that in case of parallel (disjoint) lines the coefficients of x and y in both equations must be same.
(ii) The area of parallelogram $= \frac{q_1 q_2}{\sin\theta}$, where q_1 and q_2 are length between two pairs of opposite edge and ϕ is the angle between any two adjacent edges.
It is to be noted that area of the parallelogram bounded by the four straight lines say,

$$\begin{array}{lll} y = m_1 x + c_1 & \ldots (i) & y = m_1 x + c_2 \\ y = m_2 x + d_1 & \ldots (iii) & y = m_2 x + d_2 \end{array}$$

is given by

$$z = \left| \frac{(c_1 - c_2)(d_1 - d_2)}{m_1 - m_2} \right|$$

Example . The given equations of 3 lines are

$$x + 2y + 3 = 0 - - - (1)$$
$$x + 2y - 8 = 0 - - - (2)$$

$$3x - y - 5 = 0 - - - (3)$$

Form 3 sides of two squares. Find the equation of remaining sides of these squares.
Solution: We make use of the formula to find the distance between the two parallel lines as

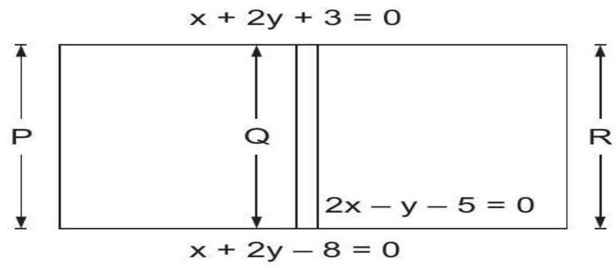
$$D = \frac{|c_1 - c_2|}{\sqrt{a^2 + b^2}}$$

Where a is the coefficient of x and b is the coefficient of y. c_1, c_2 are constant terms. Consider equations (1) and (2) to find the value of D, we have

$$D = \frac{|8 + 3|}{\sqrt{5}} = \frac{11}{\sqrt{5}}$$

The equations of sides P and R are of the form

$$2x - y + k = 0$$



Since distance between sides P and $Q =$ distance between sides Q and R

$$\frac{|k-(-5)|}{\sqrt{5}} = \frac{11}{\sqrt{5}}$$

$$\Rightarrow \quad \frac{k+5}{\sqrt{5}} = \frac{11}{\sqrt{5}} \text{ or } k+5 = 11 \text{ or } k = 11 - 5 = 6$$

$$\text{and} \quad \frac{k+5}{\sqrt{5}} = -\frac{11}{\sqrt{5}} \text{ or } k+5 = -11 \text{ or } k = -11 - 5 = -16$$

In order to obtain the fourth sides of the two squares are (i) $2x - y + 6 = 0$ and (ii) $2x - y - 16 = 0$.

Circle

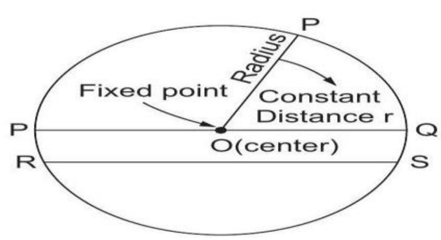
CONCEPT OF CIRCLE

A circle is a closed two -dimensional circular figure in which the set of all the points in the plane is equidistant from a given fixed point called the "centre" of the circle and the constant distance is called the radius of the circle.

Diameter of a circle $(PQ) = 2 \times r = 2r$.
Circumference of a circle $= C = 2\pi r$. It is the length of boundary of the circle.

The area of a circle is area bounded by the circumference and computed using the following formula, that is

$$\text{Area } (A) = \pi r^2$$



where π is a constant quantity and is equal to $\frac{22}{7}$ or $\pi = 3.14$ and r is the radius of the circle.
The line joining any two points on the circumference is called a chord. If the chord passes through centre then it is called diameter. In Fig. (3.9) $RS =$ chord, $PQ =$ diameter, $O =$ centre
Note: A circle is a special case of an ellipse (conic section).

General Equation of Circle

Let us write the mathematical formulation of the general circle equations as

$$x^2 + y^2 + 2gx + 2fy + c = 0 - - - - - (1)$$

where x, y are variables, g, f and c are constants and centre is $(-g, -f)$ i.e., $\left(-\frac{\text{coefficient of } x}{2}, -\frac{\text{coefficient of } y}{2}\right)$ and radius $r = \sqrt{g^2 + f^2 - c}$.
Note: If radius r is real then the circle is known as a real circle; if radius $r = 0$ then the circle is a point/ vertex circle; if radius r is imaginary then the circle is also imaginary.
Example . Find the centre and radius of the circles
(a) $2x^2 + 2y^2 - 6x - 8y + 2 = 0$
(b) $x^2 + y^2 + 2x\sin\theta + 2y\cos\theta - 10 = 0$
(c) $2x^2 + \lambda xy + 2y^2 + (\lambda - 2)x + 4y - 6 = 0$ for some λ.

Solution:
(a) We reconstruct the given equations as

$$x^2 + y^2 - \frac{6}{2}x - \frac{8}{2}y + \frac{2}{2} = 0 \Rightarrow x^2 + y^2 - 3x - 4y + 1 = 0 - -(i)$$

We derive from equation (1) the value of g, f and c, as $g = \frac{-3}{2}, f = \frac{-4}{2}, c = 1$
Hence centre is $\left(\frac{3}{2}, 2\right)$ and the radius is

$$\sqrt{g^2 + f^2 - 1} = \sqrt{\left(\frac{3}{2}\right)^2 + 2^2 - 1} = \sqrt{\frac{21}{4}} = \frac{\sqrt{21}}{2}$$

Therefore radius $r = \frac{\sqrt{21}}{2}$
(b) Consider the equation $x^2 + y^2 + 2x\sin\theta + 2y\cos\theta - 10 = 0$

Centre of this circle is $(-\sin\theta, -\cos\theta)$
Therefore, radius $= \sqrt{(-\sin\theta)^2 + (\cos\theta)^2 + 10} = \sqrt{\sin^2\theta + \cos^2\theta + 10} = \sqrt{1 + 10}$

$$(\because \sin^2\theta\cos^2\theta = 1))$$

$\therefore\ r = 3.3$
(c) Consider the given equations of the circle as

$$2x^2 + \lambda xy + 2y^2 + (\lambda - 2)x + 4y - 4 = 0 - -(ii)$$

We rewrite the equation after dividing the given equation by the coefficient of x^2 i.e., 2 we get

$$\frac{2x^2}{2}+\frac{\lambda xy}{2}+\frac{2y^2}{2}+\frac{(\lambda-2)\cdot x}{2}+\frac{4y}{2}-\frac{4}{2}=0--(iii)$$

or $x^2+\frac{\lambda}{2}\cdot xy+y^2+\frac{(\lambda-2)}{2}\cdot x+2y-2=0$
In the general equations of a circle, there is no term of xy but in our equation (ii), there is a term of xy whose coefficient is $\frac{\lambda}{2}$,
that is $\therefore\ \frac{\lambda}{2}=0 \Rightarrow \lambda=0$

So equation (iii) reduces to

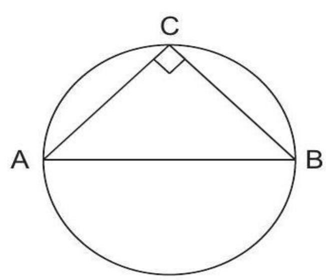
$$x^2+y^2-x+2y-2=0--(iv)$$

Hence centre is $\left(\frac{1}{2},-1\right)$ i.e., $\left(-\frac{\text{coefficient of } x}{2},-\frac{\text{coefficient of } y}{2}\right)$

Radius $=\sqrt{g^2+f^2-c}=\sqrt{\left(-\frac{1}{2}\right)^2+1^2+2}=\frac{\sqrt{13}}{2}$

Characteristics of a Circle

(i) The diameter of a circle is the longest chord of the circle and divides circle into two equal halves.
(ii) The perpendicular drawn on any given chord of a circle from the centre of the circle bisect the chord. (Perpendicular bisector)
(iii) For a given length of perimeter the circle is the shape with largest area.
(iv) The circles are said to be congruent if they have equal radii.
(v) The angle in a semi-circle is always 90°. (angle)



(vi) All circles are similar, regardless of the measure of radii or diameters.
(vii) Equal chords of a circle subtend equal angles at the centre.

Note:

1. The equation $x^2+y^2+2gx+2fy+c=0$ has three constants, hence to get the general equation of the circle at least three conditions should be known, this implies that a unique circle passes through three non collinear points.
2. A general second degree equation in x and y with variables h, g, f, c as constants is written as $ax^2+by^2+2hxy+2gx+2fy+c=0$. It represents a circle if –

- Coefficient of x^2 = coefficient of y^2 or $a = b \neq 0$
- Coefficient of $xy = 0$ implies that $h = 0$
- $(g^2 + f^2 - c) \geq 0$ (for a real circle).

Example . A pizza has a diameter of 10 inches. What is the radius of the pizza?
Solution: As diameter = 2 × radius

$$\therefore \text{ radius } = \frac{10}{2} = 5 \text{ inches Ans.}$$

Find Equation of Circle Given

I. Centre and radius.
II. Three points lying on it.
III. Coordinates of end points of diameter.

Centre and Radius

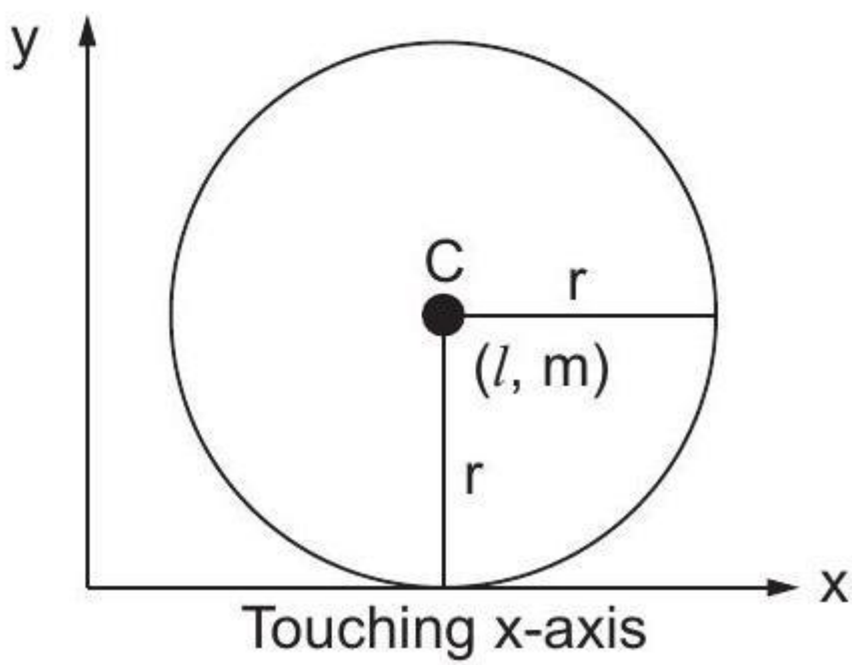
If (l, m) is the centre and r is the radius of the circle then its equation is given as

$$(x - l)^2 + (y - m)^2 = r^2 - - - - - - (1)$$

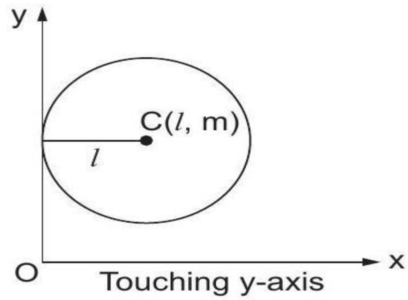
Particular cases:

(i) If centre is origin (0,0) and radius is ' r ', then equation of a circle given in (1) reduces to $x^2 + y^2 = r^2$
(ii) If radius of circle is zero, then equation of circle reduces to $(x - l)^2 + (y - m)^2 = 0$ (zero/point circle)
(iii) When circle is drawn in such a way that it touches x -axis then equation of circle is

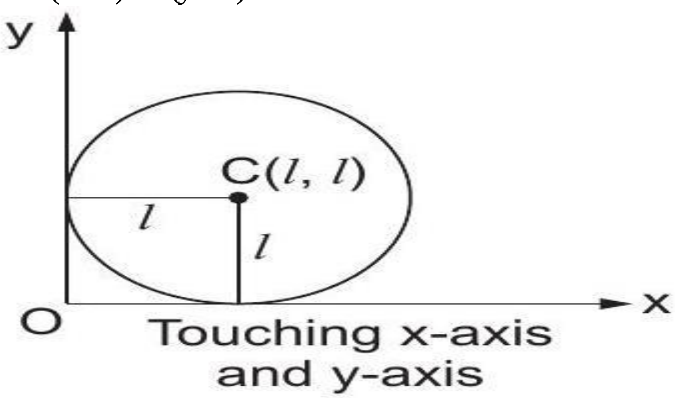
$$(x - l)^2 + (y - m)^2 = m^2 \qquad (r = m)$$



(iv) When circle is drawn in such a way that one part of it is touching the y-axis, then the equations takes the form $(x - l)^2 + (y - m)^2 = l^2$

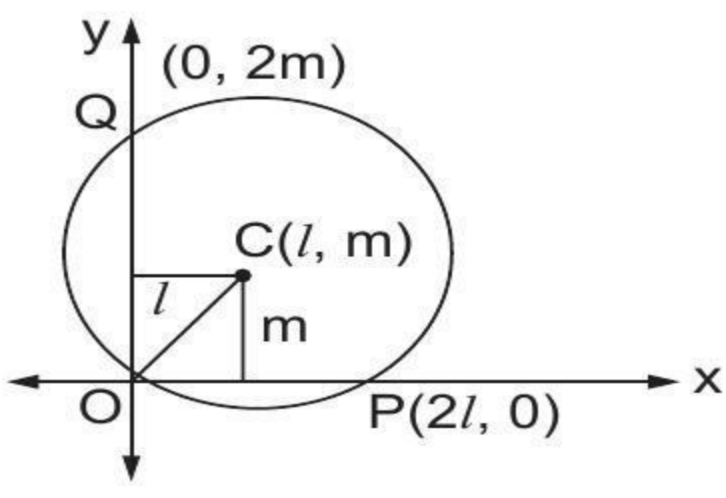


(v) When circle touches both axes (x-axis and y-axis) then equation of the drawn circle is $(x - l)^2 + (y - l)^2 = l^2$



(vi) When circle passes through the origin and centre of the circle is at a fixed point $C(l, m)$, then radius $(l^2 + m^2)^{1/2} = r$ and intercept cut on x-axis, is $OP = 2l$ and intercept cut on y-axis, is $OQ = 2m$ and hence, equation of circle is $(x - l)^2 + (y - m)^2 = l^2 + m^2$ or $x^2 + y^2 - 2lx - 2my = 0$.

It is very important to note at this stage of study that centre of the circle may lie in any quadrant, hence for general cases use ± sign before l and m, *i.e.*, the coefficients of l and m are either +1 or –1.



Note: When expanded form of circle is given we can find out its centre and radius.

Rule for finding the centre point and radius of a circle.

Consider the general equation of circle, $x^2 + y^2 + 2gx + 2fy + c = 0$

(i) Write the equation in such an algebraic form so that the coefficients of x^2 and y^2 are each equation to unity.

If it is not the case, then divide both sides of general equation by the coefficient of either x^2 or y^2, which are the same (equal) in the case of circle.

(ii) Coordinates of the centre of the circle are $\left(-\frac{1}{2}\text{ coefficient } x, -\frac{1}{2}\text{ coefficient of } y\right)$

(iii) Radius of the circle $= \left[\left(\frac{1}{2}\text{ coefficient of } x\right)^2 + \left(\frac{1}{2}\text{ coefficient of } y\right)^2 - \text{ constant term }\right]^{1/2}$

Nature of the circle derived on the basis of the values of radius of a circle.

Example . Find the equations of that diameter of the circle $x^2 + y^2 - 4x + 2y - 10 = 0$, which passes through the origin.

Solution: The coordinates of the centre C of the given circle is given by $\left[-\frac{1}{2}(-4), -\frac{1}{2}(2)\right]$ or $(2, -1)$
Therefore, required diameter is the straight line joining the origin $(0,0)$ and the centre $C(2, -1)$ and hence the required equation is obtained using the formula,

$$(y - y_1) = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

In this case $y_1 = 0, y_2 = -1, x_1 = 0, x_2 = 2$, we have

$$y - 0 = \frac{-1 - 0}{2 - 0}(x - 0) \text{ or } y = \frac{-1}{2}x \text{ or } 2y = -x \text{ or } x + 2y = 0 \text{ Ans.}$$

Example. Find the equation of the circle through $(3, -8)$ and $(-6,6)$ and having the centre on

$$2x - y = 7$$

Solution: Consider the general equation of the circle as

$$x^2 + y^2 + 2gx + 2fy + c = 0 - - - (i)$$

If this passes through $(3, -8)$, then we have

$$(3)^2 + (-8)^2 + 2g(3) + 2f(-8) + c = 0 \text{ or } 9 + 64 + 6g - 16f + c = 0$$

or

$$73 + 6g - 16f + c = 0 - - - (ii)$$

If it passes through $(-6,6)$, we have from (i)

$$(-6)^2 + (6)^2 + 2g(-6) + 2f(6) + c = 0$$
$$36 + 36 - 12g + 12f + c = 0$$

$$-12g + 12f + c = -72 \text{ -----------}(iii)$$

As the centre point $(-g, -f)$ lies on $2x - y = 7$, so, we have

$$2(-g) - (-f) = 7 \text{ or } -2g + f = 7 - - - -(iv)$$

From (ii) and (iv), we get

$$-13f + c = -52 - - - (v)$$

From (iii) and (iv), we get

$$6f + c = -114 - - - (vi)$$

Subtracting (vi) from (v), we get

$$\begin{aligned} &-13f + \not{c} = -52 \\ &-6f + \not{c} = -114 \\ &- \quad + \\ &-19f = 62 \\ &\Rightarrow = \frac{-62}{19} = -3.3 \end{aligned}$$

Therefore from (v), we get

$c = -52 - 13 \times 3.3 = -52 - 42.9 = -94.9$

From (iv), we get

$$\begin{aligned} -2g - 3.3 \quad &= 7 \Rightarrow -2g = 7 + 3.3 = 10.3 \\ \therefore g \quad &= \frac{-10.3}{2} = -5.1 \end{aligned}$$

Now we have, $f = -3.3, g = -5.1, c = -94.9$

Hence the required equation is

$$x^2 + y^2 + 2(-5.1)x + 2(-3.3)y - 94.9 = 0$$
$$x^2 + y^2 - 10.2x - 6.6y - 94.9 = 0 \text{ Ans.}$$

Equation of a Circle through Three Given Points

In order to derive the equation of a circle through three given vertices, consider three points say $L(x_1, y_1), M(x_2, y_2)$ and $N(x_3, y_3)$ and let the equation of the required circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \qquad (i)$$

Since the three points $L(x_1, y_1), M(x_2, y_2)$ and $N(x_3, y_3)$ all lie on (i), so we have,

$$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0 \qquad (ii)$$
$$x_2^2 + y_2^2 + 2gx_2 + 2fy_2 + c = 0 \qquad (iii)$$
$$x_3^2 + y_3^2 + 2gx_3 + 2fy_3 + c = 0 \qquad (iv)$$

Eliminating, g, f and c from $(i), (ii), (iii)$, and (iv) we get the required equations, which is the following structure

$$\begin{vmatrix} x^2+y^2 & x & y & 1 \\ x_1^2+y_1^2 & x_1 & y_1 & 1 \\ x_2^2+y_2^2 & x_2 & y_2 & 1 \\ x_3^2+y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0$$

Corollary: If the four points (x_r, y_r), where $r = 1,2,3,4$ lie on a circle, then

$$\begin{vmatrix} x^2+y^2 & x_1 & y_1 & 1 \\ x_2^2+y_2^2 & x_2 & y_2 & 1 \\ x_3^2+y_3^2 & x_3 & y_3 & 1 \\ x_4^2+y_4^2 & x_4 & y_4 & 1 \end{vmatrix} = 0$$

Example . Find the equation of the circle which passes through the points $L(0,1), M(1,0)$ and $N(3,2)$. Also find the value of radius and coordinates of the centre.

Solution: Suppose the general equation of a circle is given as

$$x^2 + y^2 + 2gx + 2fy + c = 0 - - - (i)$$

If it is going through the point $L(0,1)$ then

$(0)^2 + (1)^2 + 2g(0) + 2f(1) + c = 0$ or $1 + 2f + c = 0$ or $2f + c + 1 = 0 - - - (ii)$

If it passes through the point $M(1,0)$ then, we have

$$1^2 + 0^2 + 2g(1) + 2f(0) + c = 0 \text{ or } 1 + 2g + c = 0 \text{ or } 2g + c + 1 = 0 - - - (iii)$$

If it passes through the point $N(3,2)$, then, we get

$$3^2 + 2^2 + 2g(3) + 2f(2) + c = 0$$
$$9 + 4 + 6g + 4f + c = 0$$

$$6g + 4f + c + 13 = 0 - - - (iv)$$

Let us consider equation (ii) and (iii) as

$2f + c + 1 = 0$
$2g + c + 1 = 0$

From these two, we get

$$2f = 2g \text{ or } f = g - - - (v)$$

From (iv) and (v), we get

$$6g + 4g + c + 13 = 0 \text{ or } 10g + c + 13 = 0 - - - (vi)$$

Subtracting (iii) from (vi), we get

$$8g = -12 \text{ or } g = -\frac{12}{8} = -1.5 - - - (vii)$$

From (v), we get $f = g = -1.5$ and from (ii), we get

$$2(-1.5) + c + 1 = 0 \text{ or } -3.0 + c + 1 = 0 \text{ or } c = 3.0 - 1 = 2.0$$

Now, we have $f = g = -1.5, c = 2.0$, hence the required equation of the circle is obtained on putting $f = g = -1.5, c = 2.0$ in (i), we get

$$x^2 + y^2 - 2(1.5)x - 2(1.5)y + 2.0 = 0$$
$$x^2 + y^2 - 3x - 3y + 2 = 0$$

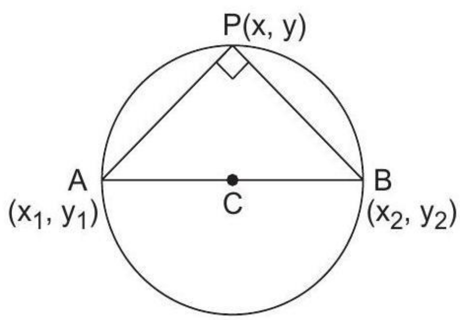
Now we have to determine the coordinates of centre c, obtained as $(-g, -f)$ or (1.5,1.5)

$$\text{And the radius } = \sqrt{g^2 + f^2 - c} = \sqrt{(1.5)^2 + (1.5)^2 - 2} = \sqrt{2.25 + 2.25 - 2}$$
$$= \sqrt{4.50 - 2} = \sqrt{2.50} = 1.6$$

Equation of Circle in Diameter Form

If $A(x_1, y_1)$ and $B(x_2, y_2)$ are the end points of the diameter of the circle and $P(x, y)$ is the point on the circumference other than A and B on the circle, then from geometrical aspect, we know that $\angle APB = 90°$ this implies that (slope of PA) $\times$ (slope of PB) $= -1$.

$$\therefore \left(\frac{y - y_1}{x - x_1}\right)\left(\frac{y - y_2}{x - x_2}\right) = -1$$
$$\Rightarrow (x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$$



Remark: This equation is a circle having least radius passing through (x_1, y_1) and (x_2, y_2).

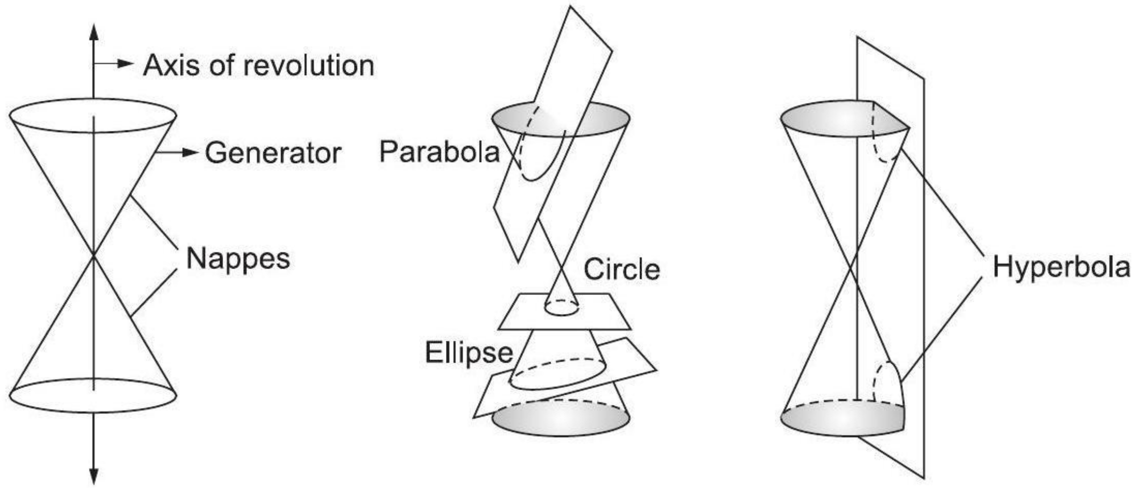
CONIC SECTIONS

Conic sections (or simply conics) are well defined curves obtained by the intersection of the surface of a cone with a plane. There are basically three types of conics

(i) Parabola

(ii) Hyperbola

(iii) Ellipse (Circle is a special case of ellipse)



A cone and conic sections

S.No.	When intersecting plane -	Then conics obtained is -
(i)	is parallel to the generating line	Parabola
(ii)	is parallel to the axis of revolution	Hyperbola
(iii)	intersects a nappe at an angle to the axis (other than 90°)	Ellipse
(iv)	is perpendicular (90°) to the axis of revolution	Circle

A conic section or conics is the locus of a point which moves in a plane so that its distance from a fixed point is in a constant ratio to its perpendicular distance from a fixed straight line and this fixed point does not lie on a fixed line. The fixed point is called the focus. The constant ratio is known as the eccentricity (denoted by). The fixed straight line is called the directrix. The line passing through its focus and perpendicular to directrix is called the axis of revolution. And point of intersection of conic with its axis of revolution is called a vertex.

General Equation of a Conic

The general equation of a conic with focus (s, t) and directrix $lx + my + n = 0$ is given as

$$(l^2 + m^2)[(x - s)^2 + (y - t)^2] = e^2(lx + my + n)$$
$$\Rightarrow \ ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

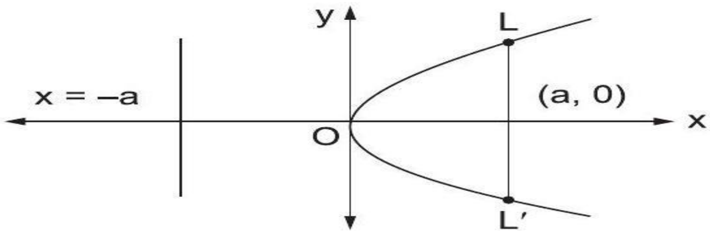
If $e > 1$, the conic is called hyperbola.
If $e = 1$, the conic is called parabola.
If $e < 1$, the conic is called ellipse.

Parabola

A parabola is a geometrical structure which is the locus of a point, which moves in a geometrical plane such that its distance from a fixed point is always equal to its distance from a fixed straight line (i.e., directrix). Standard equation of a parabola is

$$y^2 = 4ax - - - - (1)$$

where x and y are variables and a is constant.



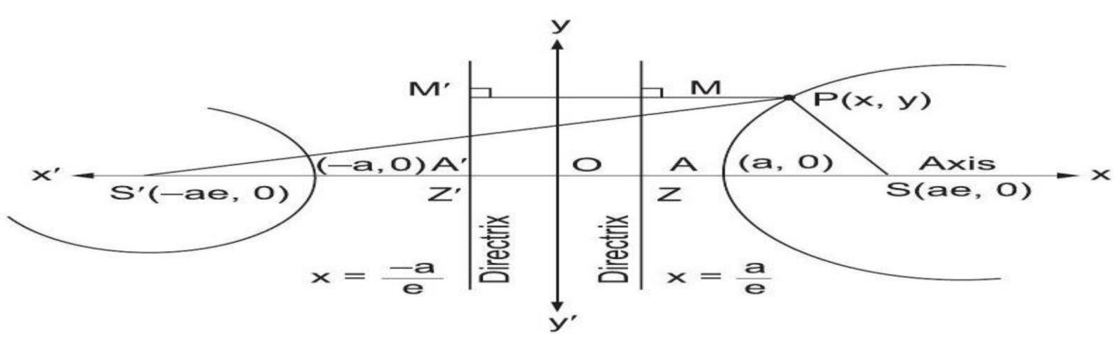
From the equation (1) of parabola, we extract the
(i) Vertex is $(0,0)$
(ii) Focus is $(a, 0)$
(iii) Axis is $y = 0$
(iv) Directrix is $x + a = 0$

Some Important Terms

(a) Focal distance: The length of a point on the parabola from its focus is called the focal distance of the point.
(b) Focal chord: A chord of the parabola which moves through its focus is called a focal chord.
(c) Latus rectum: The chord passes through focus and perpendicular to the directrix called axis of the parabola is called the latus rectum.
For parabola $y^2 = 4ax$, we derive the following terms:
(i) Length of the latus rectum $= 4a$.
(ii) Length of the semi latus rectum $= 2a$.
(iii) Ends of the latus rectum are $L(a, 2a)$ and $L'(a, -2a)$.

Hyperbola

Hyperbola is derived from conic when eccentricity $e > 1$.
Definition: The locus of a point which moves such that its length from a fixed point (focus) is e times its length from a fixed straight line (called the directrix) is defined as hyperbola. Hence hyperbola is characterized by $e > 1$.
Now we express the hyperbolic structure in mathematical formulation as follows:
The coordinate of the focus S are $(ae, 0)$ and the equation of directrix is $x = a/e$.
Let $P(x, y)$ be any point on the hyperbola,



then we have $SP/PM = e$ or $SP^2 = e^2 \times PM^2$
or

$$(x - ae)^2 + (y - 0)^2 = e^2[x - (a/e)]^2$$

or $(x - ae)^2 + y^2 = (ex - a)^2$
or $(x^2)(1 - e^2) + y^2 = a^2(1 - e^2)$
or

$$\frac{x^2}{a^2} - \frac{y^2}{a^2(e^2 - 1)} = 1 \text{ or } \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

where $b^2 = a^2(e^2 - 1)$, since in our case $e > 1$, therefore $b^2 > 0$. Hence finally the general equation of the hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ (Standard equation of hyperbola)}$$

where x and y are variables, a and b are constants and $b^2 = a^2(e^2 - 1)$. Considering the standard equation of the hyperbola, we derive the following:
(i) Since only even powers of x and y occur in this equation, so hyperbola is symmetrical about both the axes.
(ii) The hyperbola does not cut y-axis in real points whereas it cut x-axis at points $(a, 0)$ and $(-a, 0)$.
(iii) For the relations - $a \leq x \leq a$, the value of y is imaginary, that is the curve does not exist in the section $x = -a$ to $x = a$.
(iv) As x increases, y also increases i.e., the curve extends to infinity.
(v) If c is length of foci from centre then $c^2 = a^2 + b^2$ for hyperbola and $e = \frac{c}{a}$

Some Important Definitions

Foci and Directrices: Since the curve is symmetrical about y-axis, therefore there exist another focus S' at point $(-ae, 0)$. Thus it is found that in the geometrical structure of hyperbola, there are two foci $S(ae, 0)$ and $S'(-ae, 0)$. Corresponding to these foci, there are two directrices whose equations are $x = a/e$ and $x = -a/e$.

Centre: Any chord of the hyperbola through O, the mid-point of AA' will be bisected at O and therefore O is called the centre of hyperbola .

We also explain the hyperbola as the set of all points (x, y) such that the difference of the lengths from (x, y) to foci is constant. The standard from of an equation of a hyperbola centered at the origin with vertices $(\pm a, 0)$ and co-vertices ($0, \pm b$) is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Ellipse

Standard equation of ellipse referred to its principal axes along the co-ordinate axes is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a > b$ and $b^2 = a^2(1 - e^2) \Rightarrow a^2 - b^2 = a^2e^2$ where $e =$ eccentricity $(0 < e < 1)$.

Foci $F \equiv (ae, 0)$ and $F' \equiv (-ae, 0)$

(1) Vertices

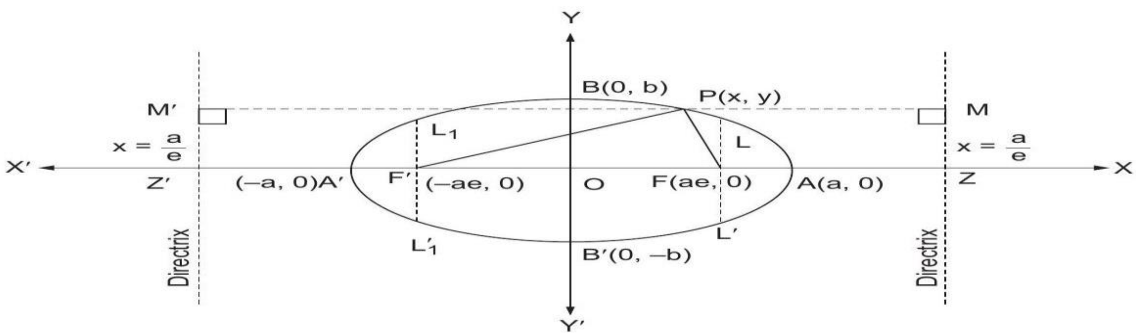
$A' \equiv (-a, 0)$ and $A \equiv (a, 0)$

(2) Equation of directrices

$x = \frac{a}{e}$ and $x = -\frac{a}{e}$

Note.The curves (viz. circles, ellipses, parabolas, hyperbolas) are known as conics because they can be obtained from a right circular cone.

(3) Major axis: The line segment $A'A$ in which the foci F' and F lie is of length $2a$ and is called the major axis $(a > b)$ of the ellipse. Point of intersection of major axis with directrix is called the foot of the directrix $\left(\pm\frac{a}{e}, 0\right)$.



(4) Minor axis: The y-axis intersects the ellipse in the points $B' \equiv (0, -b)$ and $B \equiv (0, b)$. The line segment $B'B$ of length $2b(b < a)$ is called the minor axis of the ellipse.

(5) Principal axes: The major and minor axis together are called principal axes of the ellipse.

(6) Centre: The point which bisects every chord of the conic drawn through it is called the centre of the conic $O \equiv (0,0)$. The origin is the centre of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

(7) Latus rectum: The focal chord perpendicular to the major axis is called the latus rectum.

(i) Length of latus rectum $(LL') = \frac{2b^2}{a}$ (ii) Equation of latus rectum: $x = \pm ae$

(8) Focal radii: $SP = a - ex$ and $S'P = a + ex \Rightarrow SP + S'P = 2a =$ major axis.

(9) Eccentricity: $e = \sqrt{1 - \frac{b^2}{a^2}}$

Remark: The sum of the focal lengths of any point on the ellipse is always equal to the major axis and is equal to $2a$. For all the above detailed terms, we refer Fig. 3.19.

Note:

(i) If distance of focus from the centre of ellipse $= C$ then, $C = \sqrt{a^2 - b^2}$

(ii) If the equation of the ellipse is given as $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and nothing is mentioned, then it is assumed that $a > b$.

(iii) Ellipse is symmetric with respect to both the coordinate axes.

(iv) Foci always lie on major axis.

Example . Derive the equation of an ellipse whose focus is the point $(-2,2)$, whose directrix is the line $2x-3y+5=0$ and whose eccentricity is $1/3$.
Solution: Let $P(x,y)$ be any point on the ellipse. Its focus is $S(-2,2)$ and from P let PM be the perpendicular drawn to its directrix $2x-3y+5=0$. Then $SP=ePM$ or $SP^2=e^2PM^2$

$$(x+2)^2+(y-2)^2=e^2\left[\frac{2x-3y+5}{\sqrt{(2)^2+(-3)^2}}\right]^2$$
$$\Rightarrow\ (x+2)^2+(y-2)^2=\left(\frac{1}{3}\right)^2\left[\frac{2x-3y+5}{\sqrt{13}}\right]^2=\frac{1}{(9\times 13)}(2x-3y+5)^2$$
$$\Rightarrow\ (x+2)^2+(y-2)^2=\frac{1}{117}(2x-3y+5)^2$$
$$\Rightarrow\ 117[x^2+4x+4+y^2-4y+4]=[4x^2+9y^2-12xy+20x-30y+25]$$
$$\Rightarrow 113x^2+113y^2+12xy+448x-438y+911=0$$

which is the required equation of an ellipse
Example . If latus rectum of an ellipse is $1/2$ of its minor axis, then which of the following is correct value of eccentricity ' e '
(i) $\frac{5}{2}$
(ii) $\frac{2}{5}$
(iii) $\frac{\sqrt{3}}{2}$
(iv) $\frac{2}{\sqrt{3}}$

Solution: We are given that $\frac{2b^2}{a}=\frac{2b}{2}$
($\because$ $2b$ is the value of minor axis)

$$\text{or } \frac{2b^2}{a}=b$$
$$\Rightarrow\frac{2b^2}{b}=a$$

Now squaring both sides of (i), we get

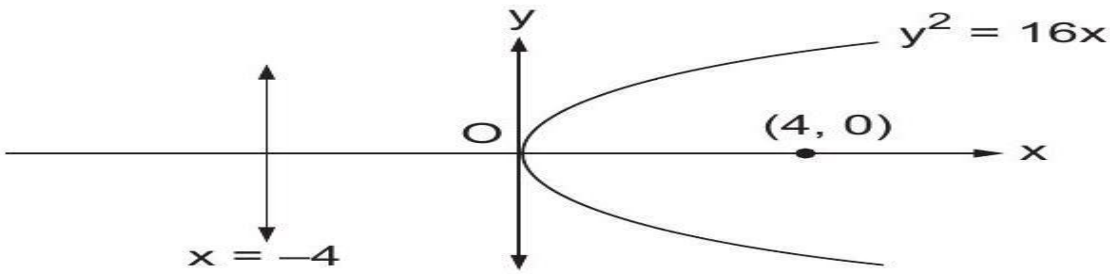
$$4b^2=a^2$$
$$\Rightarrow\quad 4a^2(1-e^2)=a^2$$
$$\Rightarrow\quad 1-e^2=\frac{1}{4}$$
$$\Rightarrow\quad e^2=1-\frac{1}{4}=\frac{3}{4}$$
$$\therefore\quad e=\sqrt{\frac{3}{4}}=\frac{\sqrt{3}}{2}$$

Ans. (iii)
Example . Find the coordinates of the focus, axis, the equation of the directrix and latus rectum of the

parabola $y^2 = 16x$.
Solution: Comparing the given equation with $y^2 = 4ax$, we get $4a = 16$

$$\Rightarrow \ a = 4$$

Thus focus of the parabola is (4,0) and the equation of the directrix of the parabola is $x = -4$.



Length of latus rectum $= 4a = 4 \times 4 = 16$.

Example. Find the equation of the parabola with focus $(4,0)$ and directrix $x = -4$.
Solution: The x-axis is the axis of parabola as focus $(4,0)$ lies on x-axis. Hence, the equation of parabola is of form $y^2 = 4ax$ or $y^2 = -4ax$. But it is given that directrix is $x = -4$ and focus $(4,0)$, the parabola must be of the form $y^2 = 4ax$ with $a = 4$
$\therefore$ required equation is $y^2 = 4 \times 4 \times x \Rightarrow y^2 = 16x$
Example . Find the coordinates of the foci, vertices, length of major axis, minor axis, eccentricity and latus rectum of the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$.
Solution: Since $9 > 4, \therefore$ the major axis is along x-axis. Comparing given equation with $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ we get get

$a = 3, b = 2$

$\therefore$ Eccentricity $e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{4}{9}} = \frac{\sqrt{5}}{3}$

$$c = \sqrt{a^2 - b^2} = \sqrt{9 - 4} = \sqrt{5}$$

POSSIBLE SHORT TYPE QUESTIONS WITH ANSWER

1. **Find the equation of the line whose x-intercept is 3 & y- intercept is 4.**

Solution: Given that a = 3 & b = 4.

So, equation of line is, $\frac{x}{3} + \frac{y}{4} = 1$

$\Rightarrow 4x + 3y = 12 \Rightarrow 4x + 3y - 12 = 0$ (Ans)

2. **Find the value of k if the lines 2x -3y+7= 0 & x-ky+2=0 are perpendicular to each other. (W-20)**

Solution: As the lines are perpendicular to each other, the product of their slopes must be equal to -1.

Let Slope of the line 2x -3y+7= 0 is $m_1 = -(2/-3) = 2/3$

and slope of the line x- ky+2=0 is $m_2 = -(1/-k) = 1/k$

As $(m_1 \times m_2) = -1 \Rightarrow 2/3 \times 1/k = -1$

$\Rightarrow k = -(2/3)$ (Ans)

3. Find the slope and y intercept of the line 2x – 3y + 8 =0

Solution: Given equation is, 2x – 3y + 8 =0

$\Rightarrow 3y = 2x + 8$

$\Rightarrow y = (2/3)x + 8/3$ Comparing this with y = mx +c

We have m = 2/3 and y- intercept = 8/3 (Ans)

Q-1 Find the equation of a circle whose end points of diameter are (-5,3) & (7,5).

Solution: The required equation of the circle is, $(x+5)(x-7) + (y-3)(y-5) = 0$

Or, $x^2 + y^2 - 2x - 8y - 20 = 0$ (Ans)

Q-2 Find the centre and radius of the circle $2x^2 + 2y^2 - 5x + 3y - 11 = 0$

Solution: First dividing the equation by 2, we get, $x^2 + y^2 - \frac{5}{2}x + \frac{3}{2}y - \frac{11}{2} = 0$

Comparing the equation with $x^2 + y^2 + 2gx + 2fy + c = 0$

we get 2g = -5/2, 2f = 3/2 and c = -11/2

or g= -5/4, f = 3/4 and c = -11/2

∴ Centre of the circle (-g, -f) i. e (5/4, -3/4) and radius of the circle is $= \sqrt{g^2 + f^2 - c}$

$$= \sqrt{\frac{25}{16} + \frac{9}{16} - \left(-\frac{11}{2}\right)} = \sqrt{\frac{25+9+88}{16}} = \sqrt{\frac{122}{16}} = \frac{\sqrt{122}}{4} \text{ units}$$

POSSIBLE LONG TYPE QUESTIONS

Q.1. Evaluate the equation of the circle which touches both the axes in first quadrant with radius 1

Q.2. If the slopes of two straight lines are 1/2 and 2, then find the angle of intersection between the two lines.

Q.3. Let *P*(5, 3) and *Q*(10, 2) be two points. Find the slope of a line perpendicular to *PQ*.

Q.4. An ellipse 9*x*2 + 25*y*2 = 225 is given. Find the foci and eccentricity.

Q.5. Find the equation of hyperbola with eccentricity 3/2 and foci at (±1, 0).

Q.6. Find the equation of parabola with directrix at $x = -6$ and focus at (6, 0).

Q.7. Obtain the equation of the circle passing through the points (3,4), (4, -3), and (-3,4)

Q.8. Find the equation of a circle whose centre is on the x-axis and the circle passes through origin and the point (4, 2**)**

Q.9. Find the equation of a circle whose centre is on the line 8x + 5y = 0 and the circle passes through the points (2, 1) and (3, 5**).**

UNIT-IV

VECTOR ALGEBRA

Learning Objectives:

- *Definition notation and rectangular resolution of a Vector.*
- *Addition and subtraction of Vectors.*
- *Scalar and Vector product of two Vectors.*
- *Simple problems related to work, moment and angular velocity.*

INTRODUCTION

Physical quantities are broadly divided as follows:

1. **Vector quantities:** They have both magnitude and direction. For examples displacement, velocity, weight, force, angular velocity, moment etc.
2. **Scalar quantities:** They have only magnitude. For example – Mass, volume, work temperature etc.

Example 1. Classify the following as scalars and vectors

(a) 10 m North West (b) 10–10 Coulomb (c) 20 km/hr. (d) 15 m/s towards East

(e) 100 Newton

Solution: (a) It is distance with direction, so is a vector.

(b) It is an electric charge, so is a scalar.

(c) It is speed, therefore a scalar.

(d) It is velocity of an object, so is a vector.

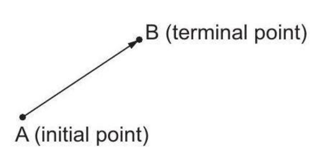
(e) It is a force, so is a vector.

Representation of Vectors

Recall: Any given part of a straight line with two end points - Initial (A) and terminal (B) is called a directed line segment (denoted as $\overrightarrow{AB}$)

A directed line segment is called a vector.

Every vector (as shown in the fig.) has the following three characteristics

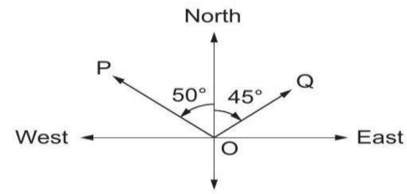


1. **Length:** The length of $\overrightarrow{AB}$ will be denoted by $|\overrightarrow{AB}|$
2. **Support:** The line of unlimited length of which vector $\overrightarrow{AB}$ is a part is called its support.

3. **Sense:** It is from its initial point to the terminal point. That is, the sense of AB is from A to B and that of BA is from B to A

Note: The vectors are denoted by putting an arrow over the symbols representing them like$\overrightarrow{AB}$,$\overrightarrow{OP}$etc. Sometimes they are also represented by a single letter $\vec{a}$,$\vec{F}$etc

Example 2. Represent graphically
(i) A displacement of $80\text{km}, 50^\circ$ west of North
(ii) A displacement of 30 km North-East

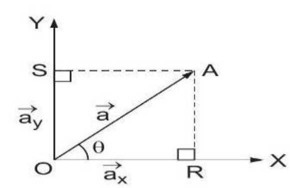


Solution: Consider the figure
(i) The vector $\overrightarrow{OP}$ represents required vector
(ii) The vector $\overrightarrow{OQ}$ represents required vector.

Note: The magnitude i.e., Distance between initial and terminal points of a vector is always a non-negative real number and is denoted as $|\overrightarrow{OA}|$ or $|\vec{a}|$ or $\boldsymbol{a}$. If $\vec{a} = x\hat{\imath} + y\hat{\jmath}$ then $|\vec{a}| = \sqrt{x^2 + y^2}$.

Rectangular Resolution of a Vector

By resolution of a vector, it means determining the effect of a vector in a particular direction and the split vectors so obtained are known as components of the vector. If these components of a given vector are perpendicular (at 90°) to each other, they are called rectangular components. Let us consider an example of finding rectangular resolution of a vector $\vec{a}$ represented by $\overrightarrow{OA}$ as shown in the figure. From the point O, two mutually perpendicular axis X and Y are drawn fromthe point A, two perpendicular AR and AS are dropped on X and Y-axis respectively.



Then consider the right-angled triangle ORA. Here the vector $\vec{a}$ makes an angle θ with position direction of X-axis. Therefore, we have $\cos\theta = \frac{OR}{OA}$.

$$\Rightarrow \qquad OR = OA\cos\theta$$
$$\Rightarrow \qquad \vec{a}_x = \vec{a}\cos\theta \qquad \dots(1)$$
$$\text{Similarly,} \qquad \vec{a}\vec{a}_y = \vec{a}\sin\theta \qquad \dots(2)$$

Thus we have resolved the vector $\vec{a}$ into two rectangular components $\vec{a}_x$ and $\vec{a}_y$ along X and Y axes respectively $\vec{a}_x$ is called the X-component of $\vec{a}$ and $\vec{a}_y$ is called the y-component of $\vec{a}$

Now, from (1) and (2) we get,

$$\cos\theta = \frac{a_x}{a} \text{ and } \sin\theta = \frac{a_y}{a}$$

$$\cos\theta = \frac{a_x}{a} \text{ and } \sin\theta = \frac{a_y}{a}$$

Squaring and adding both we get,

$$\sin^2\theta + \cos^2\theta = \frac{a_x^2}{a^2} + \frac{a_y^2}{a^2} \qquad \ldots(3)$$

But $\qquad \sin^2\theta + \cos^2\theta = 1$

$$\therefore\ (3) \quad \Rightarrow \frac{a_x^2 + a_y^2}{a^2} = 1 \quad \Rightarrow a^2 = a_x^2 + a_y^2$$

$$\Rightarrow \quad a = \sqrt{a_x^2 + a_y^2}$$

This given magnitude of $\vec{a}$ in terms of magnitudes of the components.

Note: If $\hat{\imath}$ and $\hat{\jmath}$ denote vectors of unit magnitude along OX and OY axes respectively then

$$\vec{a}_x = a\cos\theta\hat{\imath} \text{ and } \vec{a}_y = a\sin\theta\hat{\jmath}$$

So that $\vec{a} = a\cos\theta\hat{\imath} + a\sin\theta\hat{\jmath}$

Note: Unit vector in the direction of a vector $\vec{a}$ is given by $\hat{a} = \frac{\vec{a}}{|\vec{a}|}$. It has length 1.

ALGEBRA OF VECTORS

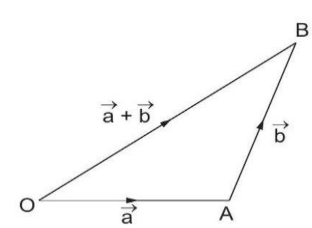
The vectors have direction together with the magnitude, so their algebra is different than that of real numbers.

(a) Addition of two vectors: Let $\vec{a}$ and $\vec{b}$ be two vectors in a plane represented by $\overrightarrow{OA}$ and $\overrightarrow{AB}$ respectively.

Then their addition can be performed in following two ways

(1) Triangle law of addition of vectors

Here $\vec{a}$ and $\vec{b}$ are two vectors to be added. Under this law we draw a figure in which the initial point of $\vec{b}$ coincides with the terminal point of $\vec{a}$. The vector joining the initial point of $\vec{a}$ with the terminal point of $\vec{b}$ is the vector sum of $\vec{a}$ and $\vec{b}$.



$$\text{i.e.,} \qquad \overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$$

$$\Rightarrow \qquad \vec{a} + \vec{b} = \overrightarrow{OB}$$

This method of addition of two vectors is called triangle law of addition of vectors.

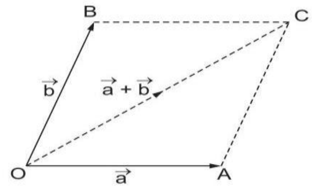
Note: Sum/resultant

If $\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath}$ and $\vec{b} = b_1\hat{\imath} + b_2\hat{\jmath}$

Then, $\vec{a} + \vec{b} = (a_1 + b_1)\hat{\imath} + (a_2 + b_2)\hat{\jmath}$

(2) Parallelogram law of addition of vector

Under this law, we draw the vectors $\vec{a}$ and $\vec{b}$ with both the initial points coinciding. Then we consider these two vectors as adjacent sides of a parallelogram. We complete the drawing of parallelogram. The diagonal so obtained from this parallelogram through the common initial point gives the sum of two vectors shown in the figure



i.e., $\overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{OC}$

$$\Rightarrow \quad \vec{a} + \vec{b} = \overrightarrow{OC}$$

This method of addition of two vectors is called parallelogram law of addition of vectors.

Properties of Vector Addition

(1) $\vec{a} + \vec{b} = \vec{b} + \vec{a}$ (Commutative)
(2) $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$ (associative)
(3) $\vec{a} + \vec{o} = \vec{a} = \vec{0} + \vec{a}$ (Additive identity)
(4) $\vec{a} + (-\vec{a}) = \vec{0} = (-\vec{a}) + \vec{a}$ (additive inverse)

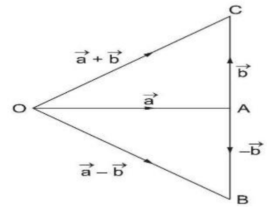
(b) Multiplication of a vector by a scalar: Let $\vec{a}$ be the given vector and m be a scalar. Then we define $\vec{b} = m\vec{a}$ as a vector of magnitude $|m\vec{a}|$. If m is positive, the direction of the vector $\vec{b} = m\vec{a}$ is same as that of $\vec{a}$ otherwise opposite to $\vec{a}$. This multiplication is called scalar multiplication. For example, if we multiply $\vec{a}$ by (-1)

then its direction gets inverted. That is, $\vec{a}$ and $-\vec{a}$ have equal magnitudes but opposite directions.

Note: If $\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath}$ then $m\vec{a} = (ma_1)\hat{\imath} + (ma_2)\hat{\jmath}$

(c) Subtraction of vectors:

Let $\vec{a}$ and $\vec{b}$ be two vectors. Then subtraction of these vectors $\vec{a} - \vec{b}$ is defined as sum of vectors $\vec{a}$ and $(-\vec{b})$. For this we invert the direction of $\vec{b}$ and add to $\vec{a}$ as shown in the figure.



Note:

1. If $\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath}$ and $\vec{b} = b_1\hat{\imath} + b_2\hat{\jmath}$ then, $\vec{a} - \vec{b} = (a_1 - b_1)\hat{\imath} + (a_2 - b_2)\hat{\jmath}$
2. Two vectors are said to be equal if they have the same magnitude/components and the same direction, regardless of their initial points. For example: If $\vec{x} = a_1\hat{\imath} + b_1\hat{\jmath}$ and $\vec{y} = a_2\hat{\imath} + b_2\hat{\jmath}$ then $\vec{x} = \vec{y}$ iff $a_1 = a_2$ and $b_1 = b_2$.

Example 3. Evaluatea, b so that the vectors $\vec{x} = a\hat{\imath} + 2\hat{\jmath}$ and $\vec{y} = 4\hat{\imath} + b\hat{\jmath}$ are equal.
Solution: Two vectors are equal if their corresponding components are equal.
Thus $\vec{x}$ and $\vec{y}$ will be equal if $a = 4$ and $b = 2$

Example 4. Let $\vec{a} = 2\hat{\imath} + 4\hat{\jmath}, \vec{b} = 4\hat{\imath} + 2\hat{\jmath}$. Is $|\vec{a}| = |\vec{b}|$? Are vectors $\vec{a}$ and $\vec{b}$ equal?
Solution: We have

$$|\vec{a}| = \sqrt{4 + 16} = \sqrt{20}$$
$$|\vec{b}| = \sqrt{16 + 4} = \sqrt{20}$$

So $|\vec{a}| = |\vec{b}|$. But the two vectors are not equal since their corresponding components are distinct.

Example 5. Find the unit vector in the direction of the sum of the vectors

$$\vec{a} = \hat{\imath} + 2\hat{\jmath} \text{ and } \vec{b} = 4\hat{\imath} + 5\hat{\jmath}$$

Solution: The sum of given vectors $\vec{a}$ and $\vec{b}$ is $\vec{c} = \vec{a} + \vec{b} = 5\hat{\imath} + 7\hat{\jmath}$

$$|\vec{c}| = \sqrt{25 + 49} = \sqrt{74}$$

Thus the required unit vector is

$$\hat{c} = \frac{\vec{c}}{|\vec{c}|} = \frac{1}{\sqrt{74}}[5\hat{\imath} + 7\hat{\jmath}]$$
$$\Rightarrow \hat{c} \quad = \frac{5}{\sqrt{74}}\hat{\imath} + \frac{7}{\sqrt{74}}\hat{\jmath}$$

Example 6. Subtract the vector $\vec{b} = 4\hat{\imath} + 9\hat{\jmath}$ from $\vec{a} = 3\hat{\imath} + 20\hat{\jmath}$.
Solution: We have to find $\vec{a} - \vec{b}$

$$-\vec{b} = -4\hat{\imath} - 9\hat{\jmath}$$
$$\therefore \ \vec{a} - \vec{b} = (3 - 4)\hat{\imath} + (20 - 9)\hat{\jmath} \ \Rightarrow \ \vec{a} - \vec{b} = -\hat{\imath} + 11\hat{\jmath}$$

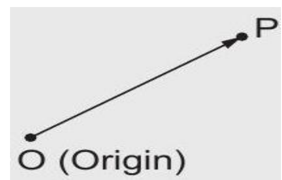
Remark: Any vector $\vec{a}$ in three dimensional systems can be expressed as $\vec{a} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$, where $\hat{\imath}, \hat{\jmath}$ and $\hat{k}$ are unit vectors parallel to x-axis, y-axis and z-axis here magnitude of $\vec{a}$ is $|\vec{a}| = \sqrt{x^2 + y^2 + z^2}$.

TYPES OF VECTORS

1. **Zero/null vector:** A vector whose magnitude is zero is called a zero or null vector and is represented by 0 .
2. **Co-initial vectors:** Two or more vectors having the same initial point are called co-initial vectors.
3. **Collinear vectors:** Two or more vectors are said to be collinear if they are parallel to the same line irrespective of their magnitudes and directions.
4. **Free vectors:** If the value of a vector depends only on its length and direction and is independent of its position in the space, it is called a free vector.
5. **Coterminous vectors:** Vectors having the same terminal points are called coterminous vectors.

Note: Position vector of a point P as shown in the figure is given as $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

Where $\hat{\imath}$ - Unit vector along x-axis
$\hat{\jmath}$ - Unit vector along y-axis
$\hat{k}$ - Unit vector along z-axis



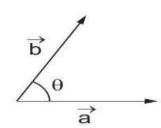
$$\therefore |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$$

PRODUCT OF TWO VECTORS

Product of two vectors is taken out by following two methods

1. Dot product or Scalar product:

The scalar product (or dot product) of two vectors $\vec{a}$ and $\vec{b}$ is defined as



$$\vec{a} \cdot \vec{b} = ab\cos\theta \quad ...(1)$$

Where a and b are the magnitudes of $\vec{a}$ and $\vec{b}$ respectively and θ is the angle between them. It is scalar quantity. Scalar product is the magnitude of $\vec{a}$ multiplied by the projection of $\vec{b}$ onto $\vec{a}(ab\cos\theta)$. The dot product between two vectors which are mutually perpendicular is zero $(\cos 90° = 0)$.

Note:

1. The scalar product is commutative i.e., $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$ and distributive i.e., $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$
2. If $\theta = 0$, then $\vec{a} \cdot \vec{b} = ab$ and $\vec{a} \cdot \vec{a} = a^2$
3. If $\theta = \pi$, then $\vec{a} \cdot \vec{b} = -ab$

4. Angle θ between $\vec{a}$ and $\vec{b}$ is given by $\cos\theta = \vec{a}\cdot\vec{b}$ (from (1))

$$\Rightarrow\ \theta = \cos^{-1}\left(\frac{\vec{a}\cdot\vec{b}}{ab}\right)$$

Scalar product in terms of components:

If $\vec{a} = a_1\hat{\imath} + b_1\hat{\jmath} + c_1\hat{k}$ and $\vec{b} = a_2\hat{\imath} + b_2\hat{\jmath} + c_2\hat{k}$

then $\quad \vec{a}\cdot\vec{b} = a_1a_2 + b_1b_2 + c_1c_2$

In particular, $\quad \vec{a}\cdot\vec{a} = |\vec{a}|^2 = a_1^2 + b_1^2 + c_1^2$

Example 7. Evaluate ' θ ' between the vectors

$$\vec{a} = \hat{\imath} + \hat{\jmath} - 2\hat{k}, \vec{b} = \hat{\imath} + 2\hat{\jmath} + 3\hat{k}$$

Solution: The angle θ between two vectors $\vec{a}$ and $\vec{b}$ is given by $\cos\theta = \frac{\vec{a}\cdot\vec{b}}{ab}$

$$\vec{a}\cdot\vec{b} = (1\cdot 1) + (1\cdot 2) + ((-2)\cdot 3) = 1 + 2 - 6 = -3$$

$$a = |\vec{a}| = \sqrt{6}$$

$$b = |\vec{b}| = \sqrt{14}$$

$$\therefore\ \cos\theta = \frac{-3}{\sqrt{2\times 3\times 7\times 2}}$$

$$\Rightarrow\ \theta = \cos^{-1}\left[\frac{-3}{2\sqrt{21}}\right]$$

Example 8. If $\vec{a} = \hat{\imath} - \hat{\jmath} + 5\hat{k}$ and $\vec{b} = \hat{\imath} + \hat{\jmath} + \lambda\hat{k}$, then find λ such that $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$ are orthogonal.

Solution: Consider $\vec{a}+\vec{b} = 2\hat{\imath} + (5+\lambda)\hat{k}$ and $\vec{a}-\vec{b} = -2\hat{\jmath} + (5-\lambda)\hat{k}$

It is given that $(\vec{a}+\vec{b})$ and $(\vec{a}-\vec{b})$ are orthogonal $(90°)$

$$\therefore\ (\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b}) = 0$$

$$\Rightarrow (2\cdot 0) + (0\cdot(-2) + (5+\lambda)(5-\lambda) = 0$$

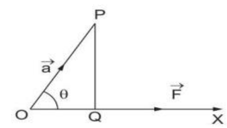
$$\Rightarrow 25 - \lambda^2 = 0$$

$$\Rightarrow \lambda = \pm 5$$

Applications of Dot Product

WORK DONE (MECHANICS)

A force acting on a particle is said to do work, if the particle is displaced in a direction which is not perpendicular to force. It is a scalar quantity.



Work done = Force × Displacement along the direction of force as shown in the figure

Work done $= \vec{F} \cdot \vec{a} = Fa\cos\theta$.

Example 9. Find the work done by the force $\vec{F} = \hat{\imath} + 2\hat{\jmath} + \hat{k}$ in the direction of displacement

$$\vec{a} = 3\hat{\imath} + 4\hat{\jmath} - \hat{k}$$

Solution: Given

$$\vec{F} = \hat{\imath} + 2\hat{\jmath} + \hat{k} \text{ and } \vec{a} = 3\hat{\imath} + 4\hat{\jmath} - \hat{k}$$

∴ Work done,

$$W = \vec{F} \cdot \vec{a} \Rightarrow W = 3 + 8 - 1 \Rightarrow W = 10 \text{ units.}$$

Example 10. A force $\vec{F} = 2\hat{\imath} + 5\hat{\jmath} - \hat{k}$ acts at a point A. The point of application of force moves from the point A to point A', where $2\hat{\imath} + 4\hat{\jmath} + 5\hat{k}$ and $3\hat{\imath} + 5\hat{\jmath} + \hat{k}$ are the position vectors of A and A' respectively. Find the work done.

Solution: Given

$$\vec{F} = 2\hat{\imath} + 5\hat{\jmath} - \hat{k}$$

Position vector of point $A = 2\hat{\imath} + 4\hat{\jmath} + 5\hat{k}$

Position vector of point $A' = 3\hat{\imath} + 5\hat{\jmath} + \hat{k}$

$$\therefore \text{ Displacement } \vec{a} = \overrightarrow{AA}$$

$$(3\hat{\imath} + 5\hat{\jmath} + \hat{k}) - (2\hat{\imath} + 4\hat{\jmath} + 5\hat{k}) = \hat{\imath} + \hat{\jmath} - 4\hat{k}$$

$$\Rightarrow \text{ Work done } W = \vec{F} \cdot \vec{a}$$

$$\Rightarrow W = 2 + 5 + 4 \Rightarrow W = 11 \text{ units}$$

Example 11. Suppose a force of 10 N acts on an object in vertically upward direction and the object is displaced through 4 m in vertically downward direction. Find the work done by the force during this displacement.

Solution: Work done $= \vec{F} \cdot \vec{a} = Fa\cos\theta$

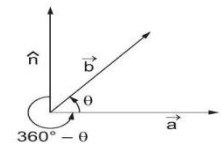
Where θ is angle between the force $\vec{F}$ and the displacement $\vec{a}$
Here

$$\theta = 180^\circ$$

Thus, $W = (10N)(4m) \cdot \cos 180^\circ$

$$= -40N - m = -40J$$

2. Cross product or vector product of two vectors



The cross product or vector product of two vectors $\vec{a}$ and $\vec{b}$, denoted by $\vec{a} \times \vec{b}$, is itself a vector defined by $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}|\sin\theta\hat{n}$. The magnitude of this vector is $|\vec{a} \times \vec{b}| = ab\sin\theta$

Where a and b are the magnitudes of $\vec{a}$ and $\vec{b}$ respectively and θ is the smaller angle between the two vectors. $\hat{n}$ is unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ (Fig. 4.10). The direction of $\vec{a} \times \vec{b}$ is perpendicular to both $\vec{a}$ and $\vec{b}$, in such a way that $\vec{a}, \vec{b}$ and this direction constitute a right-handed system.

Let

$$\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$$
$$\vec{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$$

Then,

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Note:

1. $\vec{a} \times \vec{b} = 0$ iff $\theta = 0$ i.e., $\vec{a}$ and $\vec{b}$ are parallel (or collinear) to each other.
2. If $\theta = \frac{\pi}{2}$ then $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}|\hat{n}$.
3. In terms of vector product, the angle between two vectors $\vec{a}$ and $\vec{b}$ is given by

$$\sin\theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|}$$

4. Vector product is not commutative.
5. If $\vec{a}$ and $\vec{b}$ represent adjacent sides of a triangle then its area $= \frac{1}{2}|\vec{a} \times \vec{b}|$
6. If $\vec{a}$ and $\vec{b}$ represent the adjacent sides of a parallelogram, then its area is $|\vec{a} \times \vec{b}|$.

Example 12. Find $|\vec{a} \times \vec{b}|$ if $\vec{a} = 2\hat{\imath} + \hat{\jmath} + 4\hat{k}, \vec{b} = 3\hat{\imath} + 3\hat{\jmath} + 8\hat{k}$.

Solution: $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 1 & 4 \\ 3 & 3 & 8 \end{vmatrix}$

$$= \hat{\imath}(8-12) - \hat{\jmath}(16-12) + \hat{k}(6-3)$$

$$= -4\hat{\imath} - 4\hat{\jmath} + 3\hat{k}$$

$$\therefore \ |\vec{a} \times \vec{b}| = \sqrt{(-4)^2 + (-4)^2 + 3^2} = \sqrt{16+16+9}$$

$$\vec{a} \times \vec{b} = \sqrt{41}$$

Example 13. Find a unit vector perpendicular to each of the vectors $(\vec{a} + \vec{b})$ and $(\vec{a} - \vec{b})$ where $\vec{a} = \hat{\imath} + \hat{\jmath} + 2\hat{k}, \vec{b} = 2\hat{\imath} + 3\hat{\jmath} + \hat{k}$.

Solution: We have

$$\vec{a} + \vec{b} = 3\hat{\imath} + 4\hat{\jmath} + 3\hat{k}$$

$$\vec{a} - \vec{b} = -\hat{\imath} - 2\hat{\jmath} + \hat{k}$$

Vector $\vec{c}$ which is perpendicular to $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ is $\vec{c} = (\vec{a} + \vec{b}) \times (\vec{a} - \vec{b}) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3 & 4 & 3 \\ -1 & -2 & 1 \end{vmatrix}$

$$\Rightarrow \quad \vec{c} = \hat{\imath}(4+6) - \hat{\jmath}(3+3) + \hat{k}(-6+4) \mp \vec{c} = 10\hat{\imath} - 6\hat{\jmath} - 2\hat{k}.$$

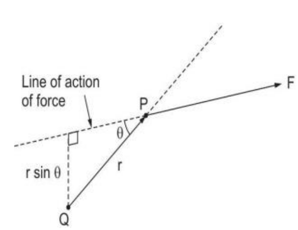
$\therefore$ Required unit vector is

$$\frac{\vec{c}}{|\vec{c}|} = \frac{5\hat{\imath}}{\sqrt{35}} - \frac{3\hat{\jmath}}{\sqrt{35}} - \frac{\hat{k}}{\sqrt{35}}$$

Applications of Vector Product

Some important applications of vector product are:

1. **Moment of force ($\vec{M}$) :** It is the rotational equivalent of linear force, also called torque ($\vec{\tau}$) or rotational force or turning effect. It is a force causing rotation around a specific point/axis, like a door rotating around its hinges. Let a force $\vec{F}$ be applied at a point P of a rigid body. Then moment of force M about a point, measures the tendency of force to turn the body about that point. If this tendency of rotation is in anti-clockwise direction, moment is positive, otherwise it is negative.



Moment of a force $\vec{M}$ about a point O is $\vec{M} = \vec{r} \times \vec{F} = \vec{\tau}$ as shown in the figure

Where $\vec{F} =$ force applied

P = point of application of force

Q = point about which we want to calculate the torque.

$\vec{r}$ = position vector of the point of application of force w.r.t the point about which we want to determine the torque.

$$|\vec{\tau}| = rF\sin\theta$$

Where θ = angle between the direction of force and the position vector P w.r.t. Q.

$r\sin\theta$ = perpendicular distance of line of action of force from point Q, it is also called force arm.

$F\sin\theta$ = component of $\vec{F}$ perpendicular to $\vec{r}$.

The moment of a force about a point is a vector quantity and is always perpendicular to the plane of rotation of the body. S.I. unit is newton-metre (N.m).

Example 14. Find the moment about (1,0,1) of the force $2\hat{\imath} + 3\hat{\jmath} + 5\hat{k}$ acting at $(2,1,-1)$.

Solution: Let

$$O \equiv (1,0,1)$$
$$A \equiv (2,1,-1)$$
$$\vec{F} = 2\hat{\imath} + 3\hat{\jmath} + 5\hat{k} \text{ (Fig. 4.15)}$$

Then, we know moment of force about O is given by $\overrightarrow{OA} \times \vec{F}$

Here

$$\overrightarrow{OA} = (2\hat{\imath} + \hat{\jmath} - \hat{k}) - (\hat{\imath} + \hat{k}) = \hat{\imath} + \hat{\jmath} - 2\hat{k}$$

$$\therefore\ \overrightarrow{OA} \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 1 & -2 \\ 2 & 3 & 5 \end{vmatrix} = \hat{\imath}(5+6) - \hat{\jmath}(5+4) + \hat{k}(3-2) = 11\hat{\imath} - 9\hat{\jmath} + \hat{k}$$

Example 15. Find the torque (moment of force) about point O and A due to given force $\vec{F}$.

Solution: Torque about point O,

$$\vec{\tau} = \vec{r_0} \times \vec{F}, \vec{r}_0 = \hat{\imath} + \hat{\jmath}, \vec{F} = 2\hat{\imath} + \hat{\jmath}$$

$$\therefore\ \vec{\tau} = (\hat{\imath} + \hat{\jmath}) \times (2\hat{\imath} + \hat{\jmath}) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 1 & 0 \\ 2 & 1 & 0 \end{vmatrix} = \hat{k}(1-2) = -\hat{k}$$

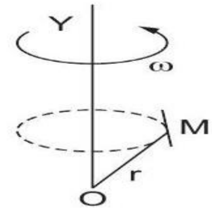
Torque about point $A, \vec{\tau} = \vec{r_a} \times \vec{F}, \vec{r_a} = \hat{\jmath}$ and $\vec{F} = 2\hat{\imath} + \hat{\jmath}$

$$\Rightarrow\ \vec{\tau} = \hat{\jmath} \times (2\hat{\imath} + \hat{\jmath})$$

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 1 & 0 \\ 2 & 1 & 0 \end{vmatrix} = -2\hat{k}$$

2. Angular velocity:

Angular velocity of an object in circular motion is defined as the rate of change of its angular displacement θ with respect to time.
In one dimension we talk about linear motion, linear displacement (x), linear velocity (v) where $v = \frac{dx}{dt}$. Similarly in two dimensions when we talk about circular motion with respect to unit time and θ as angular displacement then angular velocity, denoted by omega ω is, $\omega = \frac{d\theta}{dt}$. S.I. unit of angular velocity is radians per second.



Quite often the angular velocity is given in revolutions per second (rev/s). The conversion in radian per second may be made using $1\text{rev} = 2\pi$ radian.
Relation between linear and angular velocity: When a rigid body rotates about a fixed line OY with an angular velocity ω, then the linear velocity v of the particle M is given by $v = \omega \times r$, where $r = \overrightarrow{OM}$ (position vector of the particle with respect to O) (Fig. 4.12). $\omega = |\omega| \times$ (Unit vector along OY)
Example 16. An object has an angular speed of 4rad/s and the axis of rotation passes through the points (1,2,1) and $(2,2,-1)$. Find the velocity of the particle at point $M(2,1,3)$.
Solution: Clearly $\overrightarrow{OP} = \hat{\imath} + 2\hat{\jmath} + \hat{k}$

$$\overrightarrow{OQ} = 2\hat{\imath} + 2\hat{\jmath} - \hat{k}$$

$$\overrightarrow{PQ} = \hat{\imath} - 2\hat{k}$$

$$\Rightarrow |\overrightarrow{PQ}| = \sqrt{1+4} = \sqrt{5}$$

And $\vec{r} = \overrightarrow{PM} = (2\hat{\imath}+\hat{\jmath}+3\hat{k}) - (\hat{\imath}+2\hat{\jmath}+\hat{k}) = \hat{\imath}-\hat{\jmath}+2\hat{k}$

Now, $|\omega|$ = 4 rad/s and unit vector along $\overrightarrow{PQ} = \frac{\overrightarrow{PQ}}{|\overrightarrow{PQ}|}$

$$\therefore \quad \omega = \frac{4}{\sqrt{5}}(\hat{\imath} - 2\hat{k})$$

So, $v = \omega \times \vec{r}$

$$= \frac{4}{\sqrt{5}}(\hat{\imath} - 2\hat{k}) \times (\hat{\imath}-\hat{\jmath}+2\hat{k})$$

$$= \frac{4}{\sqrt{5}}\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 0 & -2 \\ 1 & -1 & 2 \end{vmatrix}$$

$$= \frac{4}{\sqrt{5}}[\hat{\imath}(-2) - \hat{\jmath}(4) + \hat{k}(-1)] \quad v = \frac{4}{\sqrt{5}}(-2\hat{\imath} - 4\hat{\jmath} - \hat{k}) \quad v = \frac{4}{\sqrt{5}}(2\hat{\imath} + 4\hat{\jmath} + \hat{k})$$

POSSIBLE SHORT TYPE QUESTIONS WITH ANSWER

Q.1 Find the moment about $(1,0,1)$ of the force $2\hat{\imath} + 3\hat{\jmath} + 5\hat{k}$ acting at $(2,1,-1)$.
Solution: Let

$$O \equiv (1,0,1)$$
$$A \equiv (2,1,-1)$$
$$\vec{F} = 2\hat{\imath} + 3\hat{\jmath} + 5\hat{k} \text{ (Fig. 4.15)}$$

Then, we know moment of force about O is given by $\overrightarrow{OA} \times \vec{F}$
Here

$$\overrightarrow{OA} = (2\hat{\imath} + \hat{\jmath} - \hat{k}) - (\hat{\imath} + \hat{k}) = \hat{\imath} + \hat{\jmath} - 2\hat{k}$$

$$\therefore\ \overrightarrow{OA} \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 1 & -2 \\ 2 & 3 & 5 \end{vmatrix} = \hat{\imath}(5+6) - \hat{\jmath}(5+4) + \hat{k}(3-2) = 11\hat{\imath} - 9\hat{\jmath} + \hat{k}$$

Q.2 Find $|\vec{a} \times \vec{b}|$ if $\vec{a} = 2\hat{\imath} + \hat{\jmath} + 4\hat{k}, \vec{b} = 3\hat{\imath} + 3\hat{\jmath} + 8\hat{k}$.

Solution: $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 1 & 4 \\ 3 & 3 & 8 \end{vmatrix}$

$$= \hat{\imath}(8-12) - \hat{\jmath}(16-12) + \hat{k}(6-3)$$

$$= -4\hat{\imath} - 4\hat{\jmath} + 3\hat{k}$$

$$\therefore\ |\vec{a} \times \vec{b}| = \sqrt{(-4)^2 + (-4)^2 + 3^2} = \sqrt{16+16+9}$$
$$\vec{a} \times \vec{b} = \sqrt{41}$$

Q.3 Find the work done by the force $\vec{F} = \hat{\imath} + 2\hat{\jmath} + \hat{k}$ in the direction of displacement

$$\vec{a} = 3\hat{\imath} + 4\hat{\jmath} - \hat{k}$$

Solution: Given

$$\vec{F} = \hat{\imath} + 2\hat{\jmath} + \hat{k} \text{ and } \vec{a} = 3\hat{\imath} + 4\hat{\jmath} - \hat{k}$$

$\therefore$ Work done,

$$W = \vec{F} \cdot \vec{a} \Rightarrow W = 3 + 8 - 1 \Rightarrow W = 10 \text{ units.}$$

Q.4 If $\vec{a} = \hat{\imath} - \hat{\jmath} + 5\hat{k}$ and $\vec{b} = \hat{\imath} + \hat{\jmath} + \lambda\hat{k}$, then find λ such that $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ are orthogonal.

Solution: Consider $\vec{a}+\vec{b}=2\hat{\imath}+(5+\lambda)\hat{k}$ and $\vec{a}-\vec{b}=-2\hat{\jmath}+(5-\lambda)\hat{k}$

It is given that $(\vec{a}+\vec{b})$ and $(\vec{a}-\vec{b})$ are orthogonal $(90°)$

$$\therefore\ (\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b})=0$$

$$\Rightarrow (2\cdot 0)+(0\cdot(-2)+(5+\lambda)(5-\lambda)=0$$

$$\Rightarrow 25-\lambda^2=0$$

$$\Rightarrow \lambda=\pm 5$$

Q.5 Find the torque (moment of force) about point O and A due to given force $\vec{F}$.

Solution: Torque about point O,

$$\vec{\tau}=\overrightarrow{r_0}\times\vec{F}, \vec{r}_0=\hat{\imath}+\hat{\jmath}, \vec{F}=2\hat{\imath}+\hat{\jmath}$$

$$\therefore\ \vec{\tau}=(\hat{\imath}+\hat{\jmath})\times(2\hat{\imath}+\hat{\jmath})=\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ 1 & 1 & 0\\ 2 & 1 & 0\end{vmatrix}=\hat{k}(1-2)=-\hat{k}$$

Torque about point $A, \vec{\tau}=\overrightarrow{r_a}\times\vec{F}, \overrightarrow{r_a}=\hat{\jmath}$ and $\vec{F}=2\hat{\imath}+\hat{\jmath}$

$$\Rightarrow\quad \vec{\tau}=\quad \hat{\jmath}\times(2\hat{\imath}+\hat{\jmath})$$

$$\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ 0 & 1 & 0\\ 2 & 1 & 0\end{vmatrix}=-2\hat{k}$$

POSSIBLE LONG TYPE QUESTIONS

Q.1. Position of a particle in a rectangular coordinate system is (3, 2, 5). Find out its position vector.

Q.2. If a particle moves from point P(2, 3, 5) to point Q(3, 4, 5). Find its displacement vector.

Q.3. Find the angle made by the vector A = $\hat{\imath}+\hat{\jmath}$ with x-axis.

Q.4. Prove that the three vectors $\vec{A}=3\hat{\imath}-2\hat{\jmath}+\hat{k}$, $\vec{B}=\hat{\imath}-3\hat{\jmath}+5\hat{k}$ and $\vec{C}=2\hat{\imath}+\hat{\jmath}-4\hat{k}$ form a right angled triangle

Q.5. If $\vec{A}\times\vec{B}=|\vec{A}\cdot\vec{B}|$ then can you prove that angle between $\vec{A}$ and $\vec{B}$ is equal to 45°? Give reasons.

UNIT–5
DIFFERENTIAL EQUATION

Learning Objective:

Solution of first order and first-degree differential equation by variable separation method (simple problems).

MATLAB – Simple Introduction.

DIFFERENTIAL EQUATION

You are familiar with the concept of an equation which is a mathematical statement with an "equal to" symbol between two expressions with equal values. For example: $x^2 + 2x + 1 = 0; 4\sin x + \tan x = 0, 3x + 2y = 4$ etc. In this unit, we will study equations which involve derivatives as their terms apart from variables. Such equations are called differential equations. From differential calculus you learnt how to find out derivative of a function and from integral calculus, you know how to find a function whose derivative is given. So, combining these we would study some basic concepts related to differential equations.

BASIC DEFINITIONS/CONCEPTS

A differential equation is an equation involving dependent variables, independent variables and derivatives of dependent variables with respect to one or more independent variables.

Here x is an independent variable, y is dependent variable and $\frac{dy}{dx}$ is the derivative of the dependent variable w.r.t. the independent variable.

Examples: - i) $\frac{dy}{dx} + xy = x^2$

ii) $\frac{d^3y}{dx^3} + x^4\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^4 + y = \sin x$

iii) $\frac{dy}{dx} + \sin x = \cos x.$

Differential equations are of two types as follows: -

i) Ordinary Differential Equation

ii) Partial Differential Equation

i)Ordinary Differential Equation

An ordinary differential equation is an equation involving one dependent variables, one independent variable and derivatives of dependent variable with respect to independent variable.

Mathematically $F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \dots\dots\right) = 0$

Examples: - i) $\frac{dy}{dx} + y = x^2$

ii) $\frac{d^3y}{dx^3} + x^4\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^4 + y = \sin x$

ii)Partial Differential Equation

A partial differential equation is an equation involving dependent variables, independent variables and partial derivatives of dependent variable with respect to independent variables.

Examples: - $\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}=xy$

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0$$

Order and degree of a differential equation

Order: -The order of the differential equation is the highest order of the derivatives occurring in it i.e., order of a differential equation is ' n ' if the order of the highest order derivative term present in the equation is n.

Example: $-\frac{dy}{dx}+\mathrm{y}=2x$

The highest order derivative term in the equation is $\frac{dy}{dx}$, which has order 1.

∴ order of the differential equation is 1 .

Example-: $-\frac{d^4y}{dx^4}+\frac{d^3y}{dx^3}+\left(\frac{dy}{dx}\right)^4+\mathrm{y}=\mathrm{x}$

The highest order derivative term is $\frac{d^4y}{dx^4}$, having order 4.

Hence the above differential equation has order 4.

Degree: -A degree of a differential equation is the highest exponent of the highest order derivative in the equation, after the equation has been freed from fractions and radicals as far as derivatives are concerned.

Before finding degree of a differential equation, first we have to eliminate those derivative terms present in fraction form i.e., in the denominator and derivatives with radicals i.e $\sqrt{\frac{dy}{dx}},\sqrt[3]{\frac{dy}{dx}},\sqrt[4]{\frac{dy}{dx}}$ terms. Example: - Find the order and degree of following ordinary differential equations.

i) $\frac{d^2y}{dx^2}=3\left(\frac{dy}{dx}\right)^4+xi$

Ans: - i) $\frac{d^2y}{dx^2}=3\left(\frac{dy}{dx}\right)^4+\mathrm{x}$

Here $\frac{d^2y}{dx^2}$ is the highest order derivative term.
Hence order of the differential equation is 2.
Again, equation does not contain any derivative term in fractional form or with radical.
Power of the highest order derivative term $\frac{d^2y}{dx^2}$ is 1.
Hence degree of differential equation is 1.

ii) $\frac{d^2y}{dx^2}=\sqrt{1+\left(\frac{dy}{dx}\right)^2}$

{As the above equation contain square root, so first we have to remove square root.}

Squaring both sides we have, $\left(\frac{d^2y}{dx^2}\right)^2=1+\left(\frac{dy}{dx}\right)^2$

Now $\frac{d^2y}{dx^2}$ is the highest order term with power 2.

∴ order = 2 and degree = 2.

Linear and Non-linear Differential Equation

A differential equation is said to be linear if it satisfies following conditions.

i) Every dependent variable and its derivatives have power ' 1 '.

ii) The equation has neither terms having multiplication of dependent variable with its derivatives nor multiplication of two derivative terms.

Otherwise, the equation is said to be nonlinear.

Examples: - i) $\frac{dy}{dx} + \text{xy} = x^2$

ii) $\frac{d^3y}{dx^3} + x^4 \frac{d^2y}{dx^2} + y = \sin \text{x}$

iii) $\frac{d^3y}{dx^3} + y \frac{d^2y}{dx^2} = 4x$

iv) $\left(\frac{dy}{dx}\right)^3 + 1 = 2 \frac{dy}{dx}$

v) $\frac{d^3y}{dx^3} + \frac{dy}{dx} \frac{d^2y}{dx^2} + y = 4x^2$

Among the above examples (i) and (ii) satisfy all the condition of linear equation. So the 1^{st} two equations represent linear equations.

The (iii) is nonlinear because of the term $y \frac{d^2y}{dx^2}$ which is a multiplication of dependent variable y and derivative term $\frac{d^2y}{dx^2}$.

The example (iv) is not linear due to the 1^{st} term which contain $\frac{dy}{dx}$ with power 3 violating the 1^{st} condition of linearity.

The example (v) is not linear due to 2^{nd} term which does satisfy the 2^{nd} linearity property.

Solution of an Ordinary Differential Equation

Any function/curve is called the solution of a differential equation if it satisfies the equation. [i.e., if left hand side is equal to right hand side of the equation.(L.H.S. = R.H.S.)].

The solution which consists of arbitrary constants is called the general solution (or integral or primitive) of the differential equation. Whereas a solution which is free from arbitrary constants is called a particular solution of the differential equation.

A differential equation may have a unique solution or many solutions or no solution. The general (or complete) solution of an n^{th} order differential equation will have n arbitrary constants.

There are two types of Solutions

i) General Solution

ii) particular Solution

General Solution

Example . The number of arbitrary constants in the general solution of a differential equation of fifth order

are:
(a) 0
(b) 3
(c) 4
(d) 5

Solution: Number of arbitrary constant in general solution = the order of the differential equation = 5.
Ans. (d)

Example . Find out the number of arbitrary constants in the particular solution of a differential equation of order 10 .

Solution: Number of arbitrary constants in any particular solution of a differential equation = 0
∴ Answer is zero.

Example . Verify that the function $y = A\sin x + B\cos x$ is a general solution of the differential equation $\frac{d^2y}{dx^2} + y = 0 (A, B$ are constants)

Solution: Given differential equation is $\frac{d^2y}{dx^2} + y = 0$
∴ $y = A\sin x + B\cos x$ will be a general solution of (1) if its substitution would give L.H.S. = R.H.S.
Consider, differentiating $y = A\sin x + B\cos x$
We get

$$\frac{dy}{dx} = A\cos x - B\sin x \Rightarrow \frac{d^2y}{dx^2} = -A\sin x - B\cos x \qquad (2)$$

Substituting the values of y and $\frac{d^2y}{dx^2}$ from (2) and (3) to (1) we get

$$\begin{aligned} \text{L.H.S. } &= -A\sin x - B\cos x + A\sin x + B\cos x \\ &= 0 = \text{ R.H.S. Hence verified.} \end{aligned}$$

➢ Particular Solution

The Solution obtained by giving particular values to the arbitrary constants in the general solution is called particular solution

Example: $-y = 3\cos x + 2\sin x$ is a particular Solution of differential equation $\frac{d^2y}{dx^2} + y = 0$.

Example -: Find the differential equation of the family of curves $y = e^x(A\cos x + B\sin x)$.

Sol: We are given

$$\mathrm{y} = e^x(\text{ A}\cos x + \text{B}\sin x)$$

Differentiating w.r.t. x we get

$$\frac{dy}{dx} = e^x(-\text{A}\sin x + \text{B}\cos x) + e^x(\text{ A}\cos x + \text{B}\sin x) = \mathrm{y} + e^x(-\text{A}\sin x + \text{B}\cos x)$$

$$\therefore \frac{dy}{dx} - y = e^t(-\text{A}\sin x + \text{B}\cos x)$$

Differentiation again w.r.t. x, we get

$$\frac{d^2y}{dx^2}-\frac{dy}{dx}=e^x(-\mathrm{A}\cos x-\mathrm{B}\sin x)+e^x(-\mathrm{A}\sin x+\mathrm{B}\cos x)$$

$$=-e^{\mathrm{r}}(\mathrm{A}\cos x+\mathrm{B}\sin x)+\frac{dy}{dx}-\mathrm{y}=-\mathrm{y}+\frac{dy}{dx}-\mathrm{y}$$

$\therefore \frac{d^2y}{dx^2}-2\frac{dy}{dx}+2y=0$ is the required differential equation.

Example - : Show that $\mathrm{A}x^2+\mathrm{By}^2=1$ is solution of the equation: $x\left[y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2\cdot-y\frac{dy}{dx}=0\right.$

Sol^n : Differentiating w.r.t. x the equation

$$\mathrm{Ax}^{\,2}+\mathrm{By}^2=1, \ldots\ldots\ldots\ldots \text{ (i) } \mathrm{Ax}+\mathrm{By}\frac{dy}{dx}=0$$

Differentiating w.r.t. x this equation again, we get $A+B\left[y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2\right]=0$

Eliminating A and B from (ii) and (iii), we get $x\left[y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2\right]-y\frac{dy}{dx}=0$
which is the required differential equation.

Formation of a Differential Equation Whose General Solution is Given

Let the general solution given be

$$f(x,y,a)=0--------(1)$$

where a is an arbitrary constant, x is the independent variable and y is the dependent variable. To obtain corresponding differential equation, we have to eliminate a, for which we need two equations. One is given by (1) and the other we obtain by differentiating (1) w.r.t. ' x '. As a result we obtain $g\left(x,y,\frac{dy}{dx}\right)=0$ which is the required differential equation (it represents a family of curves).

Similarly, we can extend the above method for two, three and more arbitrary constants to get the desired differential equation. (We need as many equations as the number of arbitrary constants) Here the order of the differential equation so obtained, representing a family of curves, is same as the number of arbitrary constants present in the equation corresponding to the family of curves.
Example . Form the differential equation representing the family of curves $y=ax$, where a is arbitrary constant.
Solution: Given equation is

$$y=ax--------(1)$$

Differentiating (1) w.r.t. ' x ' we get

$$\frac{dy}{dx}=a---------(2)$$

$\therefore$ (1) and (2) together give

$$y=x\frac{dy}{dx}\Rightarrow x\frac{dy}{dx}-y=0 \text{ which is the required differential equation.}$$

SOLUTION OF FIRST ORDER AND FIRST-DEGREE DIFFERENTIAL EQUATION BY VARIABLE SEPARATION METHOD

A first order and first degree differential equation can be written as

$$\frac{dy}{dx} = \phi(x, y) ----------- (1)$$

There are many methods for solving O.D.E. of type (1). Here we will be studying in detail only one method i.e., variable separation method.

If (1) can be written in the form

$$f(x)dx = g(y)dy ----------(2)$$

Then we say that the variables are separable. We solve such differential equations by integrating on both the sides i.e., solution is

$$\int f(x)dx = \int g(y)dy + c$$

where c is the constant of integration.

Example . Solve $(1 + x^2)dy = (1 + y^2)dx$.

Solution: Given differential equation is $(1 + x^2)dy = (1 + y^2)dx$.

$$\Rightarrow \frac{dy}{1 + y^2} = \frac{dx}{1 + x^2}$$

Now integrating both sides, we get

$$\int \frac{dy}{1 + y^2} = \int \frac{dx}{1 + x^2} + c \Rightarrow \tan^{-1} y = \tan^{-1} x + c$$

Ans.

Example . Evaluate: $\frac{dy}{dx} = e^{x-2y} + x^4 e^{-2y}$.

Solution: The given differential equation is

$$\frac{dy}{dx} = e^{x-2y} + x^4 e^{-2y} = (e^x + x^4)e^{-2y}$$

Now, separating the variable, we get

$$(e^x + x^4)dx = e^{2y}dy$$

Integrating both sides, we get

$$\int (e^x + x^4)dx = \int e^{2y}dy + c$$

$$\Rightarrow e^x + \frac{x^5}{5} = \frac{e^{2y}}{2} + c \text{ Ans.}$$

Example: - Solve $\frac{dy}{dx} = x^2 + 2x + 5$

Ans: $-\frac{dy}{dx} = x^2 + 2x + 5$

$\Rightarrow dy = (x^2 + 2x + 5)dx$

Integrating both sides we have,

$$=> \int dy = \int (x^2 + 2x + 5)dx$$

$$=> y = \frac{x^3}{3} + \frac{2x^2}{2} + 5x + C = \frac{x^3}{3} + x^2 + 5x + c$$

Example-: - Solve $\frac{dy}{dx} = \frac{2y}{x^2+1}$.

Ans: $-\frac{dy}{dx} = \frac{2y}{x^2+1}$

$$\Rightarrow> \frac{dy}{2y} = \frac{dx}{x^2 + 1}$$

$$=> \int \frac{dy}{2y} = \int \frac{dx}{x^2 + 1}$$

$$=> \frac{1}{2}\log_e y = \tan^{-1} x + C$$

$\Rightarrow \log_e y = 2\tan^{-1} x + K$ {2C = K is a constant as C is constant }

Example-: - Solve $\frac{dy}{dx} = x\cos x$

Ans: $-\frac{dy}{dx} = x\cos x \Rightarrow> dy = x\cos x dx$

Integrating both sides we have, $=> \int dy = \int x\cos x dx$

$$\Rightarrow y = x\int \cos x dx - \int \left\{\frac{d(x)}{dx}\int \cos x dx\right\} dx$$ integrating by parts }

$$\Rightarrow y = x\sin x - \int 1 \cdot \sin x dx = x\sin x + \cos x + C$$ (Ans)

Example-: - Solve $\frac{dy}{dx} = \sqrt{1 - y^2}$

Ans: $-\frac{dy}{dx} = \sqrt{1 - y^2}$

$$\Rightarrow \frac{dy}{\sqrt{1 - y^2}} = dx$$

$$=> \int \frac{dy}{\sqrt{1 - y^2}} = \int dx$$ { Integrating both sides }

$\Rightarrow \sin^{-1} y = x + c$. (Ans)

Example: - Solve $\sec^2 x\tan y dx + \sec^2 y\tan x dy = 0$

Ans: $-\sec^2 x\tan y dx + \sec^2 y\tan x dy = 0$

$$=> \sec^2 y \tan x dy = -\sec^2 x \tan y dx$$

$$=> \frac{\sec^2 y dy}{\tan y} = -\frac{\sec^2 x dx}{\tan x}$$

$$\Rightarrow \int \frac{\sec^2 y dy}{\tan y} = -\int \frac{\sec^2 x dx}{\tan x}$$

Let $\text{u} = \text{tany} \Rightarrow \text{du} = \sec^2 y dy$ and let $\text{v} = \tan \text{x} => \text{dv} = \sec^2 x dx$

$$\Rightarrow \int \frac{du}{u} = -\int \frac{dv}{v}$$

$$= -\ln v + \ln C => \ln u + \ln v = \ln C$$

$=> \ln uv = \ln C => \text{uv} = \text{C} => \text{tany} \tan \text{x} = \text{C}(\text{ Ans })$

$$\Rightarrow \ln u = -\ln v + \ln C => \ln u + \ln v = \ln C$$

Example 8. $\log\left(\frac{dy}{dx}\right) = a_1 x + a_2 y$.

Solution: The given differential equation is $\log\left(\frac{dy}{dx}\right) = a_1 x + a_2 y$.

$$\Rightarrow \quad \frac{dy}{dx} = e^{a_1 x + a_2 y}$$

$$\Rightarrow \quad \frac{dy}{e^{a_2 y}} = e^{a_1 x} dx \text{ (Separating the variables)}$$

Integrating both sides, we get $\int \frac{dy}{e^{a_2 y}} = \int e^{a_1 x} dx + c$

$$\Rightarrow \frac{e^{-a_2 y}}{-a_2} = \frac{e^{a_1 x}}{a_1} + c \Rightarrow \frac{e^{a_1 x}}{a_1} + \frac{e^{-a_2 y}}{a_2} + c = 0 \text{ Ans.}$$

Example 9. Solve $5xdy - 2ydx = 2x^2 dy$.

Solution: The given differential equation is $5xdy - 2ydx = 2x^2 dy$.

Separating the variables, we get

$$(5x - 2x^2)dy = 2ydx$$

$$\Rightarrow \frac{dy}{2y} = \frac{dx}{x(5-2x)} \Rightarrow \frac{dy}{2y} = \frac{1}{5}\left[\frac{1}{x} + \frac{2}{5-2x}\right]dx$$

Integrating both sides, we get

$$\int \frac{dy}{2y} = \frac{1}{5}\int \left[\frac{1}{x} + \frac{2}{5-2x}\right]dx + \log c$$

$$\Rightarrow \frac{1}{2}\log y = \frac{1}{5}\left[\log x + \frac{2}{-2}\log(5-2x)\right] + \log c$$

$$\Rightarrow \quad \log\sqrt{y} = \frac{1}{5}\left[\log\left(\frac{x}{5-2x}\right)\right] + \log c$$

$$\Rightarrow \quad \sqrt{y}$$

Example-:- Solve $\frac{dy}{dx} = \sin(\text{x} + \text{y})$

Ans :- $\frac{dy}{dx} = \sin(\text{x} + \text{y}) \left\{ \quad \text{Let x} + \text{y} = \text{z differentiating w.r.t. x}, 1 + \frac{dy}{dx} = \frac{dz}{dx} \right\}$

$$=> \frac{dz}{dx} - 1 = \sin z$$
$$\Rightarrow \frac{dz}{dx} = 1 + \sin z$$
$$=> \frac{dz}{1 + \sin z} = \text{dx}$$

Integrating both sides we have,

$$=> \int \frac{dz}{1 + \sin z} = \int dx$$
$$=> \int \frac{(1 - \sin z)dz}{(1 - \sin z)(1 + \sin z)} = \int dx$$
$$=> \int \frac{(1 - \sin z)dz}{1 - \sin^2 z} = \int dx$$
$$=> \int \frac{(1 - \sin^2)dz}{\cos^2 z} = \int dx$$
$$=> \int \left(\sec^2 z - \frac{\sin z}{\cos z}\frac{1}{\cos z}\right) dz = \int dx$$
$$=> \int (\sec^2 z - \tan z \sec z)dz = \int dx$$
$$=> \tan z - \sec z = \text{x} + \text{C}$$
$$=> \tan(\text{x} + \text{y}) - \sec(\text{x} + \text{y}) - \text{x} = \text{C}$$

Example-: - Find the particular solution of $\frac{dy}{dx} = \cos^2 y,\ \text{y} = \frac{\pi}{4}$ when $\text{x} = 0$.

Ans: $\frac{dy}{dx} = \cos^2 y$

$$\Rightarrow> \frac{dy}{\cos^2 y} = \text{dx} \ \Rightarrow \sec^2 y dy = dx$$

Integrating both sides we have,

$$\Rightarrow> \int \sec^2 y dy = \int dx$$

$\Rightarrow \tan y = x + C - \cdots -$ (1) (general Solution)

Now putting $x = 0$ and $y = \frac{\pi}{4}$ in equation (1) we have,

$\Rightarrow \tan\frac{\pi}{4} = 0 + C \ \Rightarrow C = 1$

From (1) and (2) we have,

$\tan y = x + 1$ (Ans)

Example:- Find the particular solution of $(1 + x)ydx + (1 - y)xdy = 0$, Given $y = 2$ at $x = 1$.

Ans:-

$$(1 + x)y\,dx + (1 - y)x\,dy = 0$$
$$=> (1 - y)x\,dy = -(1 + x)y\,dx$$
$$=> \frac{1 - y}{y}dy = -\frac{(1 + x)}{x}dx$$
$$=> \left(\frac{1}{y} - 1\right)dy = -\left(\frac{1}{x} + 1\right)dx$$
$$=> \int \left(\frac{1}{y} - 1\right)dy = -\int \left(\frac{1}{x} + 1\right)dx$$
$$= \quad \log y - y = -(\log x + x) + C \text{ (general solution)}$$

Putting $x = 1$ and $y = 2$ in Equation(1) we have,

$$=> \log 2 - 2 = -(\log 1 + 1) + C$$
$$=> \log 2 - 2 = -(0 + 1) + C = -1 + C$$
$$=> C = \log 2 - 1 - - - - - - (2)$$

From (1) and (2) we have,

$$\log y - y = -\log x - x + \log 2 - 1$$
$$=> \log y - y + \log x + x = \log 2 - 1$$

MATLAB - SIMPLE INTRODUCTION

The abbreviation MATLAB stands for MATrix LABoratory. MATLAB is a high level multi-paradigm language meant for technical and mathematical computing. It was created by Cleve Moler, initially as a teaching tool for mathematics in the 1970's. Later in 1980's MATLAB was released as a commercial product. It has an interactive environment with hundreds of built-in-functions for technical computations, graphical multidomain simulations, animations etc. MATLAB is used all over the world especially by engineers, scientists, technicians to analyze data, develop algorithms and create models.

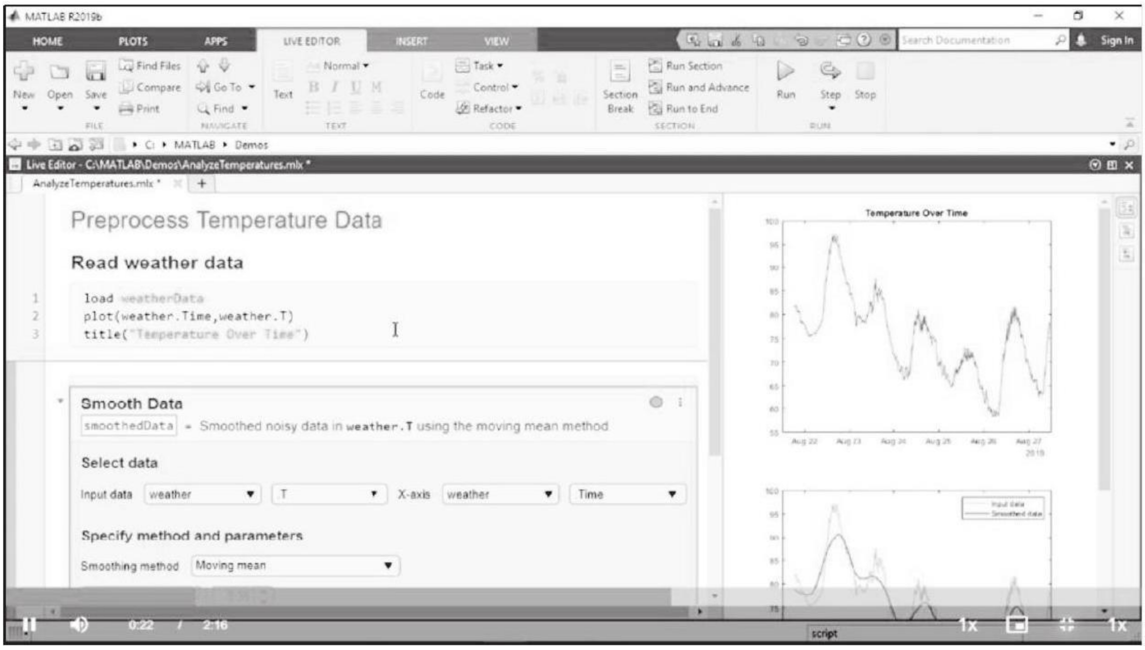
Logo of MATLAB

Salient Features

The MATLAB logo is a threedimensional L-shaped membrane. It is an eigenfunction of the wave equation. (The wave equation is a fundamental model in mathematical physics that describes how a disturbance travels through matter.)

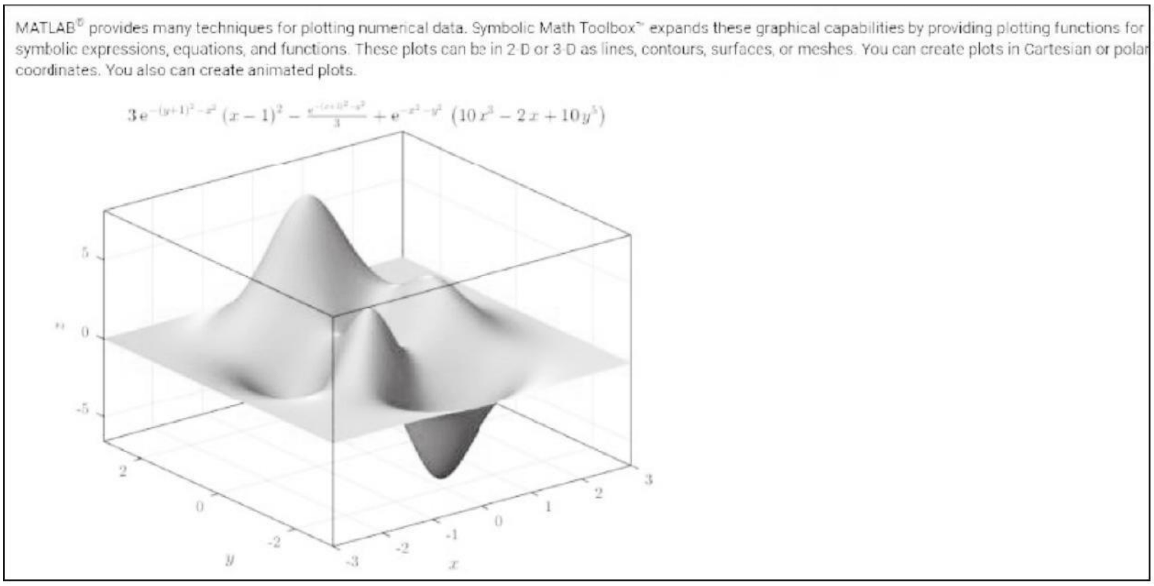
Some of the salient features of MATLAB are-

- Basic building block of MATLAB is the matrix.
- Live Editor: It has live editor for creating scripts that combines code, output and formatted text in an executable notebook.



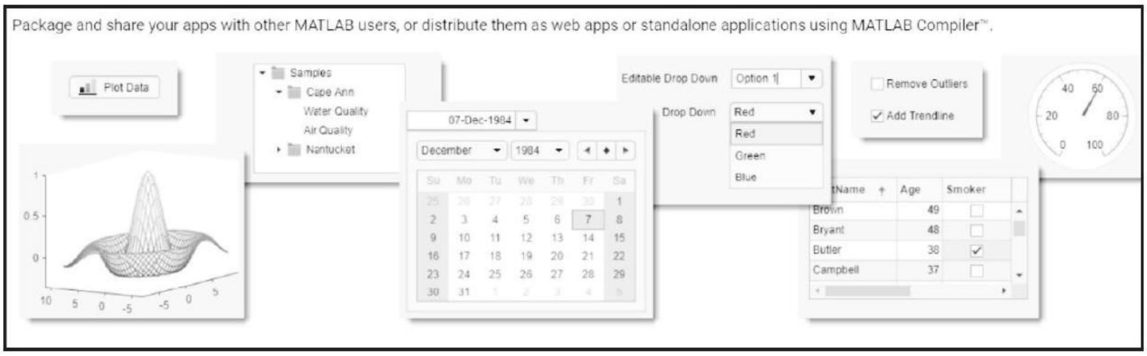
(Live Editor)

- Platform Independent: It is supported by windows, Macintosh, Linux, Unix etc.
- Program written on any of these platforms in MATLAB will run on the other too.
- Its toolboxes are professionally developed and fully documented.
- MATLAB has interactive apps through which we can iterate our data until we get desired results, thereafter generating MATLAB programs automatically related to our work.
- It scales one's analyses and there is no need to rewrite code or learn big data programming (or memorize complicated techniques.)
- Data Analysis: Thousands of prebuilt functions in MATLAB can be used to organize, clean and analyze complex date sets (which can be imported too!) in diverse fields like finance, medical etc. This can thereafter be documented using MATLAB Live Editor and exported in PDF, MS Word, Latex and HTML formats.
- Graphics: We can visualize the data by using built-in plots in MATLAB. This helps in identifying the underlying patterns and trends. These can be exported and shared too!



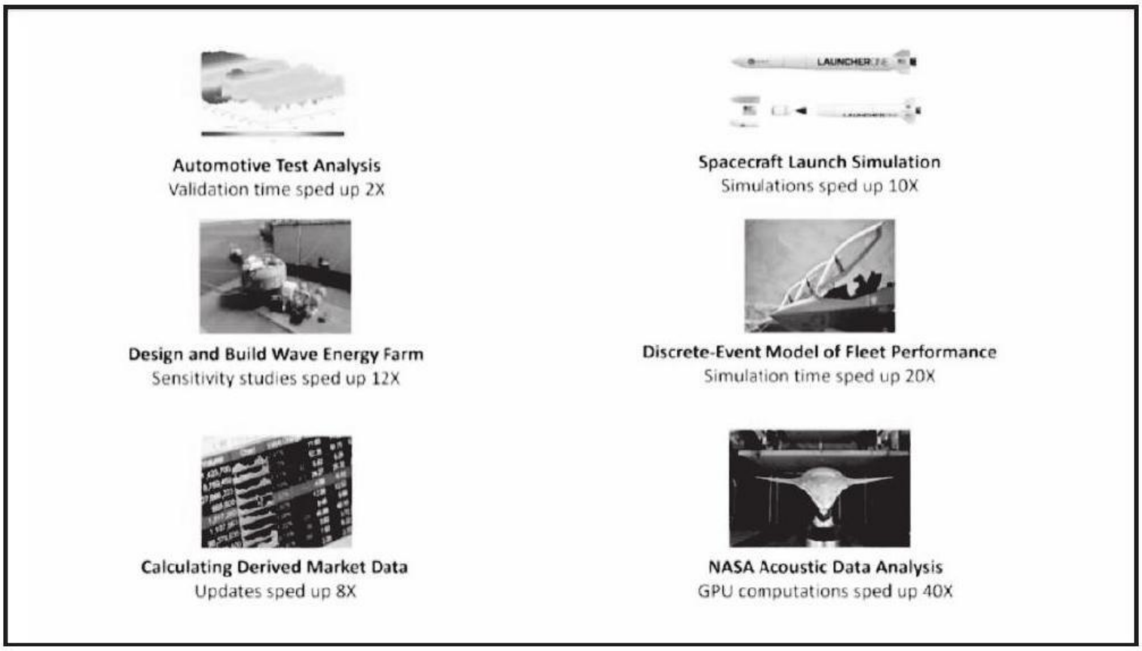
(Graphics)

- Algorithm development: It provides us the tools to transform our ideas into algorithms much faster than other languages like C, C++ or Fortran. These algorithms can be tested & verified, shared & distributed as well as deployed in larger systems.
- App Building: App designer in MATLAB allows one to create professional app (desktop & web apps) without being a professional software developer. These apps can be(royalty free) shared too!



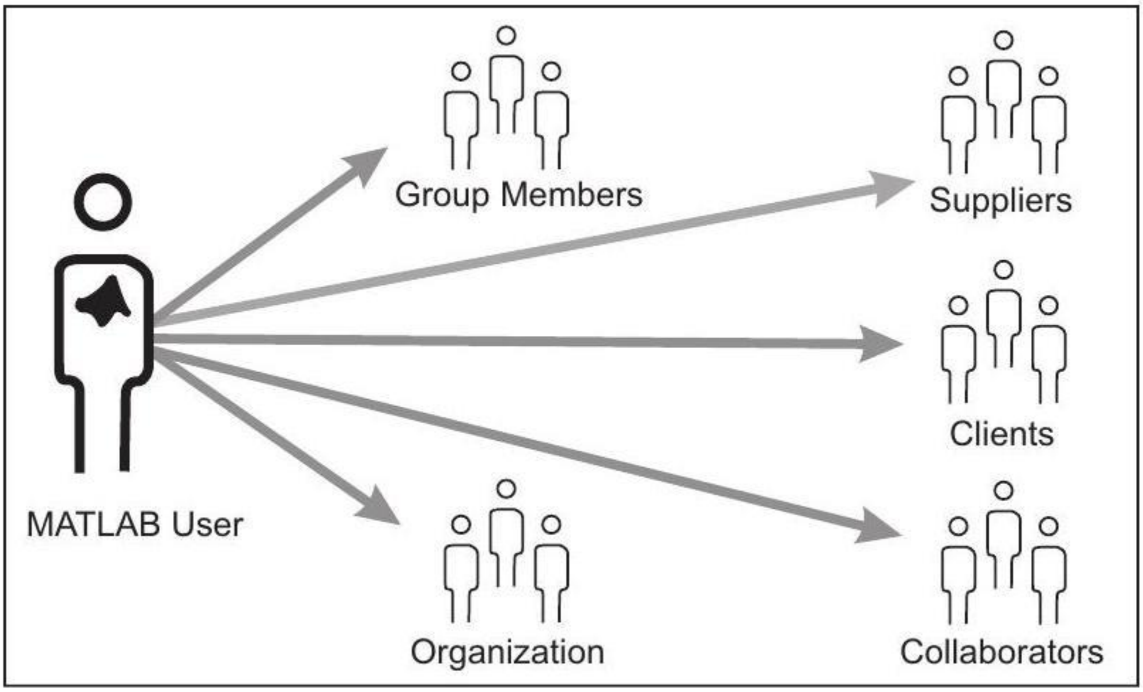
(App Building)

- MATLAB can be used with other languages too, like C,C++, Fortran, Java, Python, COM components and applications (.NET) etc. This feature helps in team work, as teams using different programming languages can work together.
- Parallel computing: This toolbox helps in performing large-scale computations using multicore desktops, graphic processing units (GPU's), computer clusters. Simulations which took months, run via this tool in a few days.



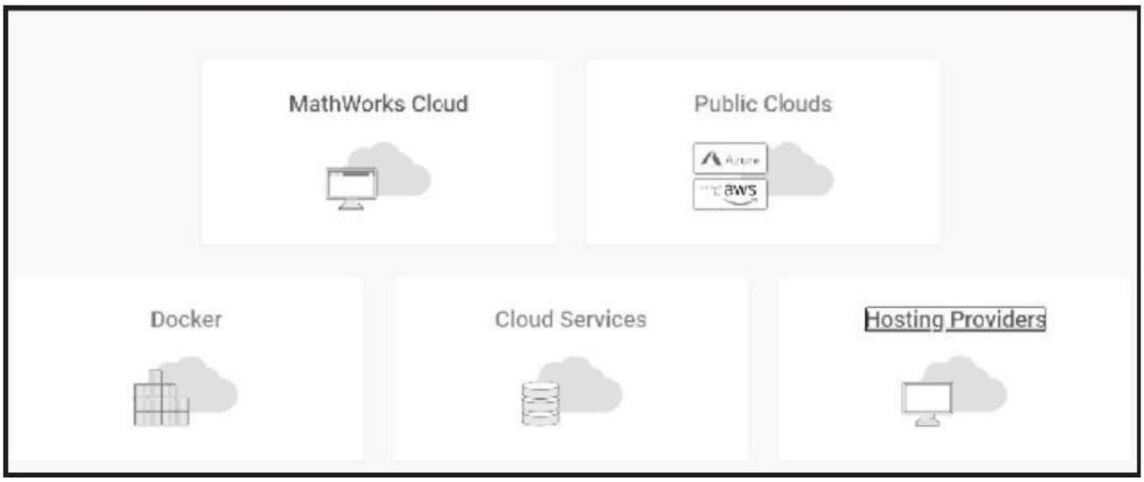
(A few parallel computing toolboxes)

- Web and desktop deployment: This application helps to share work one does in MATLAB with people who do not have access to MATLAB.



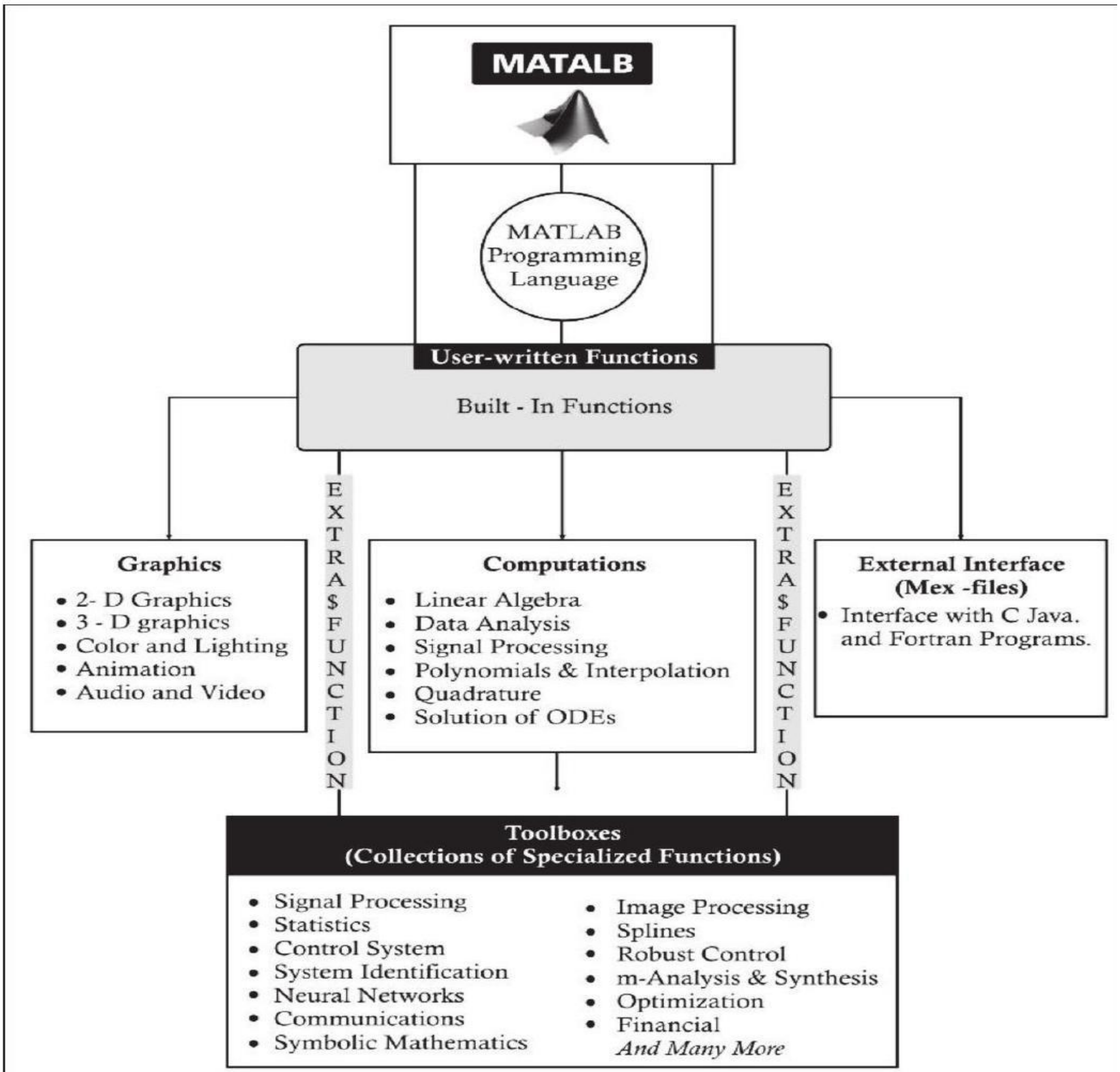
(Use of application deployment to share MATLAB programs)

- Use MATLAB in the cloud: One can use MATLAB in a web browser without installing, configuring or managing any software. MATLAB drive helps to store, access and work with one's files from anywhere.



(Runs in various cloud environments)

- A figure of some features of MATLAB is give below:

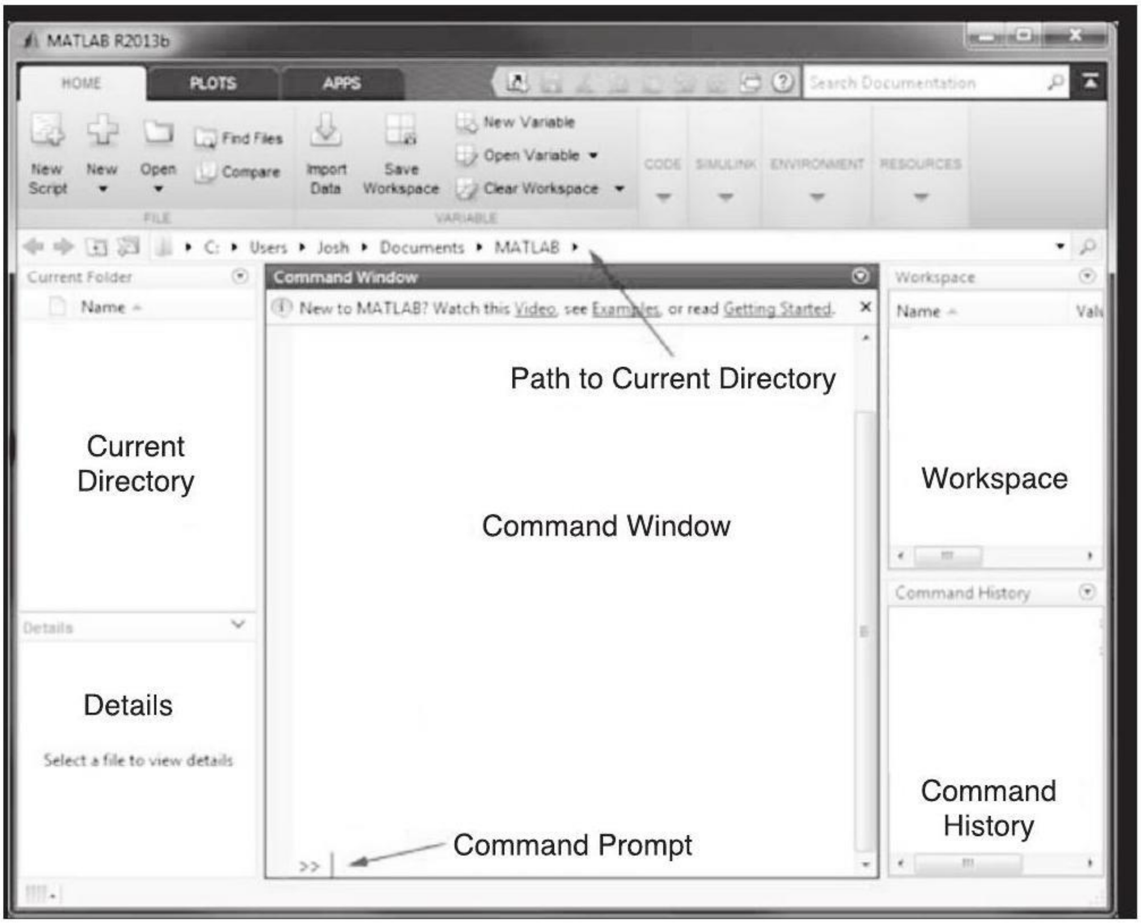


A schematic diagram of MATLAB's main features.

Basics OF MATLAB

On nearly all platforms, MATLAB works through following basics:-

1. MATLAB desktop: It is the main place where we work. It consists of the following sub windows:
   (a) Command Window: It is the main window and all commands are typed in this window at the MATLAB prompt(>>).
   (b) Current Directory Pane: Here all the files from current directory are listed.
   (c) Details (File) Pane: It is below the current directory pane and shows details of file selected in current directory pane.
   (d) Workspace Pane: It lists all variables generated by the user.
   (e) Command History Pane: All commands types in command window get recorded here.



2. Figure Window: The output of all graphics commands typed in command window are stored here.
3. Editor Window: Here the user can write, edit, create and save programs in files called M-files.
4. ON-LINE HELP: MATLAB has help option for all its function and also has demonstration programs too for explaining its features.
5. Input-Output: It supports interactive computation. The fundamental data type in MATLAB is an array/matrix. Mentioning dimensions of matrix is not needed. MATLAB is case-sensitive. The output of every command is shown on the screen unless it is directed otherwise.
6. File Types: MATLAB reads and writes several types of files. We study here following five types
(i) M-Files: They have .m extension and are of two types: (a) Script files and (b) Function files. Most of the programs are saved as M-files.
(ii) Mat-Files: They have .mat extension. These files are created by MATLAB, when user saves data with the 'save' command.
(iii) Fig-Files: They have .fig extension. These are created by saving a figure in this format.
(iv) P-Files: They have .p extension. These are compiled M-files.
(v) Mex-files: They have .mex extension.
7. Quitting MATLAB: To end MATLAB session, type quit in the command window or select the file "Exit"

MATLAB in the desktop main menu.
8. For elaborate details on MATLAB see the official website of Mathworks.

Remark: Toolboxes of MATLAB: These are a collection of numerous functions built on the MATLAB computing environment. Like - curve fitting toolbox, Create 2D plots, Fourier Transforms etc.
Example 10. Given an example of adding scalar to an array.
Solution: Create an array A and add scalar value 4 to it.

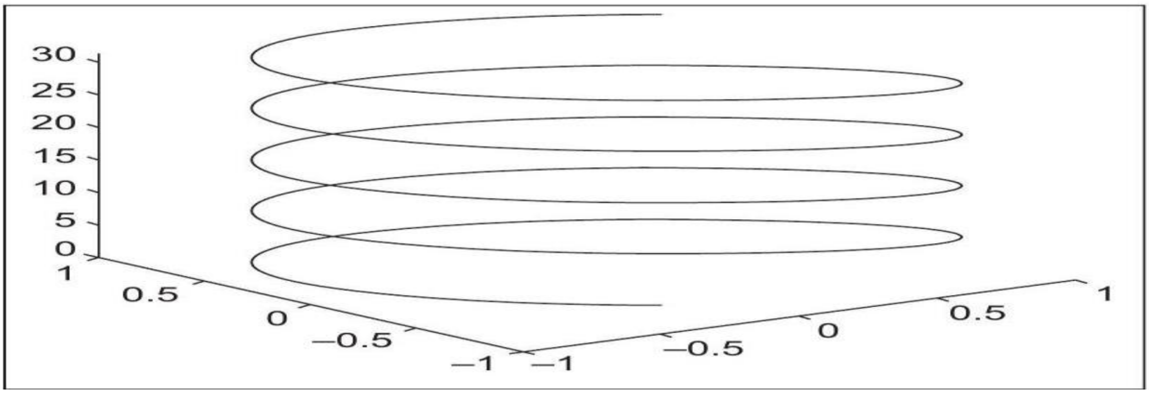
$$\gg A = [0,2;2,0]$$
$$\gg C = A + 4$$
$$C = 4\ 6$$
$$6\ 4$$

The scalar value is added to each entry of A.
Example 11. Give an example of appending strings.
Solution: Create two 1 by 3 string arrays and then append similarly located strings in the arrays.

$\gg a1$ = ["White" "Black" "Brown"]
$a1 = 1 \times 3$ string
"White" "Black" "Brown"
$\gg a2$ = ["Flower" "Vase" "Table"]
$a2 = 1 \times 3$ string
"Flower""Vase" "Table"
$\gg a = a1 + a2$
$a = 1 \times 3$ string
"White Flower" "Black Vase" "Brown Table"

Example 12. Plot 3-D Helix.
Solution: Define t as a vector of values between 0 and 10π. Define st and ct as vectors of sine and cosine values. Then plot st, ct, and t as follows-

$$\gg t = \theta; pi/50: 10^{*}pi;$$
$$\gg st = \sin(t);$$
$$\gg ct = \cos(t);$$
$$\gg \text{plot3}(st, ct, t)$$



(3-D Helix)

Advantages of MATLAB

- It can be used in multiple ways. Like- from processing still images and creating simulation videos, to calculating, to programing and so on.
- It has predefined function which makes it easy to use.

- It is supported on many different platforms.
- It has outstanding numerous tools for visualizing technical aspects.
- It has tools which allows the user to interactively design a Graphical User Interface (GUI) for user's program.
- It's memory management is automatic.
- It has ability to call external libraries.
- In MATLAB dimension statements, pointers are not required.
- It has good online tutorials.

Disadvantages of MATLAB

- It is meant for technical and mathematical computing. So, it is not applicable for other fields.
- It is an interpreted language and so at times may execute more slowly than complied language. But it can be checked through proper structuring of program.
- For large computing it requires fast computer with adequate memory.

A Few Keyboard Shortcuts for MATLAB

Action	Keyboard Shortcut
Move to the next visible panel.	Ctrl + Tab
Move to the previous visible panel.	Ctrl + Shift + Tab
Move to the next tab in a panel.	Ctrl + Page Down
Move to the previous tab in a panel.	Ctrl + Page Up

Action	Keyboard Shortcut
Make an open tool the active tool.	- Command Window: Ctrl+0 - Command History: Ctrl+1 - Current Folder: Ctrl+2 - Workspace: Ctrl+3 - Profiler: Ctrl+4 - Figure Palette: Ctrl + 6 - Plot Browser: Ctrl+7 - Property Editor: Ctrl+8 - Editor: Ctrl+Shift+0 - Figures: Ctrl+Shift+1 - Web browser: Ctrl+Shift+2 - Variables Editor: Ctrl+Shift+3 - Comparison Tool: Ctrl+Shift+4 - Help browser: Ctrl+Shift+5 On macOS systems, use the Command key instead of the Ctrl key.

Cancel the current action.	Esc (escape) For example, if you click the name of the Edit menu, the whole menu appears. Pressing Esc hides the menu again. In the Function Browser, pressing Esc up to three times has the following effects: 1. Dismiss the search history. 2. Clear the search field. 3. Close the Function Browser.

Application of Differential Equations and MATLAB

1. Sometimes, even in diverse problems of distinct scientific fields identical differential equations are obtained. This explains us the unifying principle behind diverse phenomena. Likepropagation of light and sound in the atmosphere and of waves on the surface of a pond-all can be described by the same 2nd order partial differential equation.
2. Radioactive decay modelling, population growth, prey-predator models etc. uses concept of differential equations.
3. Differential equation are used in physics, computer graphics and vision, gaming features, robotics, to predict change in investment return over time, bank interest, flow problems, seismic waves; in the field of medicine like for modelling cancer growth or the spread of disease etc.
4. Differential equations are also used with Newton's second law of motion and the law of cooling related to the temperature of objects and its surrounding.
5. MATLAB has many applications in industry as well as in academia. It is used in the software components of washing machines, printer, automobiles, industrial machines etc. With the push of one button, MATLAB generates code and runs the hardware.
6. MATLAB is such that with just a few simple lines of coding, a user can build models without having to be an expert.
7. Technical professionals use MATLAB to study big data to gain insights.
8. Apart from above, MATLAB has many applications in wide area like-robotics, computational biology, computational finance, mechatronics etc.

Note:A village has a population of 1000 people. The government launches a scheme to make all the villagers computer literate. For this, one villager is chosen to be the leading person, by making him/her computer literate. If computer literacy spread is proportional to the product of the number of computer-literate villagers and remaining villagers, and there are 100 computer-literate people after 10 days- then, answer the following questions:

Q.1. If $c(t)$ denotes the number of computer literate students at any time then, maximum and minimum value of $c(t)$ respectively is
(a) 50 and 2
(b) 100 and 1
(c) 1000 and 1
(d) 0

Q.2. The value of $c(10)$ is
(a) 50
(b) 100
(c) 1000
(d) 10

Q.3. Can you make a mathematical model related to computer literacy of the population, based on the given information? If yes, then get it verified by your teacher.

SHORT QUESTIONS WITH ANSWER

Q(1) Find order and degree of the differential equation $\left(\frac{dy}{dx}\right)^2 + \frac{1}{\frac{dy}{dx}} = 2$ [2007(S)

Sol^n : $\left(\frac{dy}{dx}\right)^2 + \frac{1}{\frac{dy}{dx}} = 2$

As $\frac{dy}{dx}$ is present in the denominator of 2nd term in L.H.S. , so we have to remove it first.

Multiplying both side by $\frac{dy}{dx}$ we have,

$$\left(\frac{dy}{dx}\right)^3 + 1 = 2\frac{dy}{dx}$$

Now the only derivative term $\frac{dy}{dx}$ has power 3 .

Hence order = 1 and degree = 3.

Q(2) Find order and degree of the differential equation $\frac{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}{\frac{d^2y}{dx^2}} = \sqrt[3]{\frac{d^2y}{dx^2}}$ (2014-S, 2016-S, 2017-W

Sol^n : $\frac{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}{\frac{d^2y}{dx^2}} = \sqrt[3]{\frac{d^2y}{dx^2}}$

The equation contain both fractional form as well as radicals, so we have to remove it.

1st multiplying $\frac{d^2y}{dx^2}$ on both sides we have,

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt[3]{\frac{d^2y}{dx^2}}\frac{d^2y}{dx^2}$$

Now squaring both sides we have,

$1 + \left(\frac{dy}{dx}\right)^2 = \left(\frac{d^2y}{dx^2}\right)^{\frac{2}{3}}\left(\frac{d^2y}{dx^2}\right)^2$ Again taking cube of both sides we have

$$\left(1 + \left(\frac{dy}{dx}\right)^2\right)^3 = \left(\frac{d^2y}{dx^2}\right)^2\left(\frac{d^2y}{dx^2}\right)^6 = \left(\frac{d^2y}{dx^2}\right)^8$$

From above the highest order derivative term $\frac{d^2y}{dx^2}$ has power 8 .

Hence order = 2 and degree = 8.

Q (3) Write the solution of $\sqrt{4 + \frac{dy}{dx}}$ =2 (2014-S)

Sol^n : $\sqrt{4 + \frac{dy}{dx}}$ =2

$\Rightarrow\ 4 + \frac{dy}{dx} = 4$

$\Rightarrow\ \frac{dy}{dx} = 0 \Rightarrow \int dy = 0 \Rightarrow\ y = c(\text{Ans})$

Q (4) Solve $\frac{dy}{dx} = \log x$ (2016-S)

Sol^n : $\frac{dy}{dx} = \log x$

$\Rightarrow dy = \log x \, dx$

$\Rightarrow \int dy = \int \log x \, dx \Rightarrow y = \int (\log x).1 \, dx + c$

$\Rightarrow y = \log x \,. x - \int \frac{1}{x} . x \, dx + c$

$\Rightarrow y = x\log x - \int dx + c$

$\Rightarrow y = x\log x - x + c$(Ans)

Q (5) How many arbitrary constants does the general solution of differential equation
$\frac{d^2y}{dx^2} = \sin x + \cos x$ (2016-S)

Sol^n : The general solution contains two arbitrary constants because the differential equation is of second order.

POSSIBLE LONG TYPE QUESTIONS

Q.1.Solve $(1 + y^2)dx + (1 + x^2)dy = 0$ (2008-S, 2010-S, 2013-S)

Q.2.Solve the differential equation $\frac{dy}{dx} + \sqrt{\frac{1-x^2}{1-y^2}} = 0$ (2008-S)

Q.3. Explain the basics of MATLAB.

Q.4. What is MATLAB. Write its merits and demerits.

Q.5. Explain order and degree of a differential equation. Given examples.

Q.6. Solve: $(1 - x)dy - (1 + y)dx = 0$

Q.7. Find the general solution of the differential equation
$x\sqrt{1 + y^2}dx + y\sqrt{1 + x^2}dy = 0$

Q.8. Find the particular solution of the differential equation
$\frac{dy}{dx} = e^{x-y} + x^2e^{-y}$ when $x = 0$ and $y = 1$